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ASYMPTOTIC BEHAVIOR OF SOLUTIONS OF DIFFERENTIAL-DIFFERENCE EQUATIONS

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Summary

In this paper we consider the problem of determining the asymptotic behavior of solutions of linear differential-difference equations whose coefficients possess asymptotic series.

Although the problem is considerably more complicated than the corresponding problem for ordinary differential equations, by means of a sequence of transformations we are able to reduce the problem to a form where the standard techniques of ordinary differential equation theory can be employed.

We first transform the differential-difference equation into an integral equation, then transform this integral equation into an integro-differential equation. At this point the Liouville transformation plays a vital role. Although the guiding ideas are simple, the analysis becomes formidable. For this reason, we have considered only some of the more immediate aspects of the problem.

ASYMPTOTIC BEHAVIOR OF SOLUTIONS OF DIFFERENTIAL-DIFFERENCE EQUATIONS

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1. Introduction

The asymptotic behavior of the solutions of linear systems of differential equations of the form

$$(1.1) \qquad x'(t) = A(t)x(t),$$

where $x(t)$ is a vector of dimension N and $A(t)$ is a given matrix of dimension N with known behavior as $t \rightarrow +\infty$, has long been a subject of investigation, and an extensive literature now exists[1]. One of the most interesting cases is that in which $A(t)$ possesses an asymptotic power series expansion,

$$(1.2) \qquad A(t) \sim \sum_{n=0}^{\infty} A_n t^{-n}.$$

For example, it is known that if the matrix A_0 has simple characteristic roots $\lambda_1, \lambda_2, \ldots, \lambda_N$, then with each root λ_j there is associated a solution $x_j(t)$ having an asymptotic expansion of the form

$$(1.3) \qquad x_j(A) \sim e^{\lambda_j t} t^{r_j} \sum_{n=0}^{\infty} c_n t^{-n}, \quad (c_0 = 1),$$

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[1] For a thorough discussion and further references, refer to Bellman, [1], Coddington and Levinson, [5].

where r_j is dependent on A_1 and where the c_n are constant vectors. Furthermore, since these N solutions are linearly independent, every solution of the equation in (1.1) is a linear combination of these particular solutions. In the case in which the characteristic roots are not all simple, similar, but more complicated results exist. Also, similar, but less precise, results are known if the relation in (1.2) is replaced by the weaker hypothesis

$$(1.4) \qquad A(t) = A_0 + A_1(t) + A_2(t),$$

where

$$\int^{\infty} \|A_1(t)\| \, dt < \infty,$$

$$\int^{\infty} \|A_2'(t)\| \, dt < \infty;$$

cf. [1], [4], [7].

A corresponding theory for differential-difference equations of the form

$$(1.5) \qquad x'(t) = \sum_{i=0}^{m} A_i(t)x(t - \omega_i)$$

has been slow to develop, though a start was made in a paper by Cooke, [6]. Asymptotic behavior of the solutions of particular equations of the form

$$(1.6) \qquad x'(t) = \sum_{i=0}^{m} (tA_i + B_i)x(t - \omega_i),$$

where the A_i and B_i are constant, has been studied by Yates, [8]. These two papers both contain quite involved analysis. As a preliminary to the present paper, one of the authors has devised an elementary method, cf. [3], for discussing the equation in (1.5) in case the $A_i(t)$ have asymptotic power series expansions and all characteristic roots of A_0 are simple.

We shall present here a new technique which will enable us to show that to each root of the characteristic exponential polynomial there corresponds a solution possessing an asymptotic expansion. The technique is applicable to differential-difference equations of the form in (1.5) under hypotheses like those in either (1.2) or (1.4). In case the hypotheses are of the type in (1.2) and all characteristic roots are simple, the expansion of the solution will be of the form in (1.3). If the roots are not simple, it will be of a form described below. The method is also applicable to ordinary differential equations of the form (1.1).

There are essentially two aspects of the problem discussed here. First, one must establish the existence of a solution associated with each characteristic root and having an asymptotic expansion of the indicated type. Second, one must prove that every solution is a linear combination of the special ones found so that we know the asymptotic behavior of every solution. For differential equations, since there are only a finite number of characteristic roots, the second proof is trivial, but for differential-difference

equations, for which there are infinitely many characteristic roots, this is not the case. In the present paper, we shall consider only the first aspect of the problem.

Throughout this paper, we shall deal only with the scalar equation

$$(1.7) \qquad u'(t) + a(t)u(t) + b(t)u(t - \omega) = 0,$$

rather than with the vector-matrix equation in (1.5), in order to hold the details, which are occasionally onerous, within reasonable limits. However, the method to be used needs no essential modification in order to be applied to the equation in (1.5). Moreover, so that the fundamental ideas of our method will be as clear as possible, we shall divide the discussion into several parts, beginning with the simplest case and taking up successively more complicated cases. In §2, we shall summarize various known results, which we need later, concerning differential-difference equations with constant coefficients. In §§3-7, we shall show how to find the asymptotic expansion associated with a simple characteristic root of the equation in (1.7), and in subsequent sections we shall extend the method to include multiple roots. We shall also indicate some generalizations to more general linear functional equations, and to nonlinear differential-difference equations. These, however, will not be discussed in detail.

2. Differential-difference Equations with Constant Coefficients

In this section, we shall summarize various known results[1] concerning the scalar equations with constant coefficients,

(2.1) $$u'(t) + a_0 u(t) + b_0 u(t - \omega) = 0.$$

The characteristic function of this equation is the exponential polynomial

(2.2) $$h(s) = s + a_0 + b_0 e^{-\omega s}.$$

The characteristic roots of the equation in (2.1), zeros of $h(s)$, lie asymptotically along the curve

(2.3) $$\mathrm{Re}(s) + \log |s| = \log |b_0|,$$

spaced an asymptotic distance of $2\pi\omega^{-1}$ apart. This curve is symmetric to the real axis, and is similar to an exponential curve for large $|s|$. As $|s| \longrightarrow \infty$, with s on the curve, the curve becomes more and more nearly parallel to the imaginary axis, and $\mathrm{Re}(s) \longrightarrow -\infty$. The roots are either real or else occur in conjugate pairs (assuming a_0 and b_0 are real), and on any vertical line there lie at most two roots. The roots with non-negative imaginary parts can be put into a sequence $\{\lambda_n\}$, $n = 1,2,\ldots,$ where $\mathrm{Re}(\lambda_n) > \mathrm{Re}(\lambda_{n+1})$.

[1] For further details, refer to Bellman, [2], or Wright, [9], [10].

Finally, all roots are simple, except possibly that $\lambda = -1 - a_0$ is a double root if the following relation holds:

$$(2.4) \qquad b_0 \omega e^{\omega(1+a_0)} = 1.$$

The equation in (2.1) is satisfied by any sum of the form

$$(2.5) \qquad u(t) = \sum_r p_r(t) e^{\lambda_r t},$$

where $\{\lambda_r\}$ is any sequence of characteristic roots, $p_r(t)$ is a polynomial of degree less than the multiplicity of λ_r, and the sum is either finite or is infinite with suitable conditions to insure convergence. (Roots of multiplicity greater than two are possible for more general equations than (2.1). The methods presented below apply to these more general situations also).

Conversely, let $\{\lambda_n\}$ be the sequence of roots described above. Then any solution of the equation in (2.1) can be represented by a series

$$(2.6) \qquad u(t) = \sum_{n=1}^{\infty}{}' e^{\lambda_n t} p_n(t),$$

wherein $p_n(t)$ is a suitable polynomial of degree less than the multiplicity of λ_n. The prime on the sum indicates that a term involving a root λ_n with posit ve imaginary part is to include both $e^{\lambda_n t} p_n(t)$ and its conjugate, which arises from the conjugate root.

The asymptotic behavior of a solution of the equation in (2.1) is fully described by the series representation in (2.6). With each characteristic root λ, of multiplicity m, there are associated m solutions, $e^{\lambda t}$, $te^{\lambda t}$, ..., $t^{m-1}e^{\lambda t}$. Conversely, every solution can be represented as an infinite linear combination of these solutions.

The present paper can be regarded as a generalization of the content of this last paragraph to differential-difference equations with asymptotically constant coefficients. We propose to show that with each root λ of multiplicity m there are associated m solutions of known asymptotic form. From the above discussion, we see that there is one real root, or a conjugate pair of complex roots, having a real part which exceeds the real parts of all other roots. This root (or each root of the conjugate pair) will be called the <u>principal</u> <u>root</u>. It turns out that the demonstration is particularly simple for a real principal root, and we shall accordingly discuss this situation first, later extending the method to the more complicated cases.

Of fundamental importance to us is the fact that a continuous solution of the equation

$$(2.7) \qquad u'(t) + a_0 u(t) + b_0 u(t - \omega) = f(t)$$

can be represented by means of an integral operator in terms of the forcing function $f(t)$ and the values of $u(t)$ over any interval of length ω, as follows:

$$(2.8)\qquad u(t) = u(t_0)k(t-\omega) - b_0\int_{t_0-\omega}^{t_0} u(t_1)k(t-t_1-\omega)dt_1$$

$$+\int_{t_0}^{t} f(t_1)k(t-t_1)dt_1, \quad t > t_0.$$

Here $k(t)$ is the unique function with these properties:

(2.9) (i) $k(t) = 0, \quad -\omega \leq t < 0;$

(ii) $k(0) = 1;$

(iii) $k(t)$ is continuous for $t \geq 0$, and $k'(t)$ is continuous for $t \geq \omega;$

(iv) $k'(t) + a_0 k(t) + b_0 k(t-\omega) = 0, \quad t > 0.$

This result is readily established by use of the Laplace transform. Moreover, $k(t)$ has a series expansion

$$(2.10)\qquad k(t) = \sum_{n=1}^{\infty}{}' e^{\lambda_n t} q_1(t),$$

where each $q_n(t)$ is a polynomial of degree less than the multiplicity of λ_n. Finally, for any positive integer N,

$$(2.11)\qquad k(t) = \sum_{n=1}^{N}{}' e^{\lambda_n t} q_n(t) + k_1(t),$$

where

$$(2.12)\qquad |k_1(t)| = O(e^{(\text{Re }\lambda_N - \varepsilon)t}) \quad \text{as} \quad t \to \infty, \quad (\varepsilon > 0).$$

We shall include here several lemmas which will be useful in what follows.

Lemma 1. If $w(t)$ is positive and non-decreasing, $u(t) \geq 0$, $v(t) \geq 0$, and all three functions are continuous, and if

$$(2.13)\qquad u(t) \leq w(t) + \int_a^t u(t_1)v(t_1)dt_1, \quad a \leq t \leq b,$$

then

$$(2.14)\qquad u(t) \leq w(t) \exp\left(\int_a^t v(t_1)dt_1\right), \quad a \leq t \leq b.$$

To prove this lemma, observe that

$$\frac{u(t)}{w(t)} \leq 1 + \int_a^t \frac{u(t_1)v(t_1)}{w(t)}dt_1 \leq 1 + \int_a^t \frac{u(t_1)v(t_1)}{w(t_1)}dt_1,$$

since w is non-decreasing. Let

$$r(t) = \int_a^t \frac{u(t_1)v(t_1)}{w(t_1)}dt_1.$$

Then

$$(2.15)\qquad u(t) \leq w(t)(1 + r(t))$$

and

$$r'(t) = \frac{u(t)v(t)}{w(t)} \leq v(t)(1 + r(t)).$$

It follows that

$$(2.16) \qquad 1 + r(t) \leq \exp\left(\int_a^t v(t_1)dt_1\right).$$

Combining (2.15) and (2.16), we get the desired result.

The remaining lemmas will be used to estimate the magnitude of various integrals which appear in our discussions. Most of them are proved by integration by parts, and all are elementary in character. We have set them apart as lemmas, and collected them in one place, for ease of reference.

Lemma 2. Suppose that $c_1 > 0$ and that $f(t)$ and $g(t)$ are real functions for which

$$\int^{\infty} |f(t)|\, dt < \infty, \quad \int^{\infty} |g(t)|\, dt < \infty.$$

Then

$$(2.17) \qquad \int_{t_0}^{t} g(t_1)\exp\left\{c_1 t_1 + \int_{t_0}^{t_1} f(t_2)dt_2\right\}dt_1$$

$$= o(1)\exp\left\{c_1 t + \int_{t_0}^{t} f(t_2)dt_2\right\} \quad \text{as} \quad t \to \infty.$$

Denote the integral in the left member of (2.17) by $J(t_0, t)$. Then

$$|J(t_0, \tfrac{t}{2})| \leq c\int_{t_0}^{t/2} |g(t_1)|\, e^{c_1 t_1} dt_1$$

$$\leq c e^{c_1 \frac{t}{2}} \int_{t_0}^{\infty} |g(t_1)|\, dt_1$$

$$= e^{c_1 t} o(1) = o(1)\exp\left\{c_1 t + \int_{t_0}^{t} f(t_2)dt_2\right\}.$$

Also

$$\left|J(\tfrac{t}{2},t)\right| \leq c\int_{t/2}^{t} |g(t_1)|\, e^{c_1 t_1} dt_1 \leq c e^{c_1 t} \int_{t/2}^{t} |g(t_1)|\, dt_1$$

$$= e^{c_1 t} o(1) = o(1)\exp\left\{c_1 t + \int_{t_0}^{t} f(t_2)dt_2\right\}.$$

<u>Lemma 3. Suppose that</u> $c_1 > 0$ <u>and that</u> $f(t)$ <u>and</u> $g(t)$ <u>are real, twice differentiable functions satisfying these conditions</u>:

(2.18) $f(t)$ <u>and</u> $g(t) \to 0$ <u>as</u> $t \to \infty$,

(2.19) $g(t) \neq 0$ <u>for</u> $t \geq t_0$,

(2.20) $g'(t) = o(g(t))$ <u>as</u> $t \to \infty$,

(2.21) $\int^{\infty} |f'(t)|\, dt < \infty$,

(2.22) $\int^{\infty} f(t)^2 dt < \infty$,

(2.23) $\int^{\infty} \left|\frac{g''(t)}{g(t)}\right| dt < \infty$,

(2.24) $\int^{\infty} \left|\frac{f(t)g'(t)}{g(t)}\right| dt < \infty$.

<u>Then</u>

(2.25) $$\int_{t_0}^{t} g(t_1)\exp\left\{c_1 t_1 + \int_{t_0}^{t_1} f(t_2)dt_2\right\} dt_1$$

$$= (c_1^{-1} + o(1))g(t)\exp\left\{c_1 t + \int_{t_0}^{t} f(t_2)dt_2\right\}, \quad \text{as} \quad t \to \infty.$$

The conditions are satisfied, if, for example, $f(t) = g(t) = t^{-a}$ where $a > 1/2$. To prove the lemma, we first note that by (2.20),

$$\lim_{t \to \infty} \frac{\log|g(t)|}{t} = 0. \tag{2.26}$$

If we integrate by parts, we obtain

$$\begin{aligned} (2.27) \quad & \int_{t_0}^{t} g(t_1)\exp\left\{c_1 t_1 + \int_{t_0}^{t_1} f(t_2)dt_2\right\}dt_1 \\ & = c_1^{-1} g(t)\exp\left\{c_1 t + \int_{t_0}^{t} f(t_2)dt_2\right\} + \text{constant} \\ & - c_1^{-2}\left[g'(t) + f(t)g(t)\right]\exp\left\{c_1 t + \int_{t_0}^{t} f(t_2)dt_2\right\} \\ & + c_1^{-2}\int_{t_0}^{t} (g'' + gf' + 2g'f + gf^2)\exp\left\{c_1 t_1 + \int_{t_0}^{t_1} f(t_2)dt_2\right\}dt_1. \end{aligned}$$

Since $c_1 > 0$, $f(t) \to 0$ as $t \to \infty$, and $g(t) = e^{t \cdot o(1)}$,

$$g(t)\exp\left\{c_1 t + \int_{t_0}^{t} f(t_2)dt_2\right\} \to \infty \quad \text{as} \quad t \to \infty.$$

Since $g'(t) = o(g(t))$, it follows that the first three terms in the right member of (2.27) can be written as

$$c_1^{-1} g(t)\exp\left\{c_1 t + \int_{t_0}^{t} f(t_2)dt_2\right\}(1 + o(1)).$$

Furthermore, denoting the last term in (2.27) by $J(t_0, t)$, we have

$$|J(t_0,\tfrac{t}{2})| \leq \exp\left\{c_1 \frac{t}{2} + \int_{t_0}^{t/2} f(t_2)dt_2\right\} \int_{t_0}^{\infty} |g'' + gf' + 2g'f + gf^2|\,dt_1$$

$$\leq g(t)\exp\left\{c_1 t + \int_{t_0}^{t} f(t_2)dt_2\right\}\exp\left\{c - \log|g(t)| - c_1 \frac{t}{2} - \int_{t/2}^{t} f(t_2)dt_2\right\}$$

$$\leq g(t)\exp\left\{c_1 t + \int_{t_0}^{t} f(t_2)dt_2\right\}o(1).$$

Also, since $g(t)\exp\left\{c_1 t + \int_{t_0}^{t} f(t_2)dt_2\right\}$ is increasing for large t,

$$|J(\tfrac{t}{2},t)| \leq g(t)\exp\left\{c_1 t + \int_{t_0}^{t} f(t_2)dt_2\right\} \cdot \int_{t/2}^{t} \left(\left|\frac{g''}{g}\right| + |f'| + 2\left|\frac{fg'}{g}\right| + f^2\right)dt_1$$

$$\leq g(t)\exp\left\{c_1 t + \int_{t_0}^{t} f(t_2)dt_2\right\}o(1).$$

This proves the relation in (2.25).

<u>Lemma 4. Suppose that</u> $c_1 > 0$ <u>and that</u> $f(t)$ <u>and</u> $g(t)$ <u>are real functions for which</u>

$$\int^{\infty} |f(t)|\,dt < \infty, \quad \int^{\infty} |g(t)|\,dt < \infty.$$

<u>Then</u>

$$\int_t^\infty g(t_1)\exp\left\{-c_1t_1+\int_{t_0}^{t_1} f(t_2)dt_2\right\}dt_1$$

$$= o(1)\exp\left\{-c_1t+\int_{t_0}^{t} f(t_2)dt_2\right\} \quad \underline{\text{as}} \quad t\to\infty.$$

For

$$\left|\int_t^\infty g(t_1)\exp\left\{-c_1t_1+\int_{t_0}^{t_1} f(t_2)dt_2\right\}dt_1\right| \leq ce^{-c_1t}\int_t^\infty |g(t_1)|\,dt_1$$

$$\leq o(1)\exp\left\{-c_1t+\int_{t_0}^{t} f(t_2)dt_2\right\}.$$

<u>Lemma 5. Suppose that</u> $c_1 > 0$ <u>and that</u> $f(t)$ <u>and</u> $g(t)$ <u>are real, twice differentiable functions satisfying the hypotheses of Lemma 3. Then</u>

$$\int_t^\infty g(t_1)\exp\left\{-c_1t_1+\int_{t_0}^{t_1} f(t_2)dt_2\right\}dt_1$$

$$= (-c_1^{-1}+o(1))g(t)\exp\left\{-c_1t+\int_{t_0}^{t} f(t_2)dt_2\right\}, \quad \underline{\text{as}} \quad t\to\infty.$$

To prove this, we observe that for t sufficiently large,

$$\left|g(t)\exp\left\{-c_1t+\int_{t_0}^{t} f(t_2)dt_2\right\}\right| \leq ce^{-c_1t/2} = o(1),$$

by (2.26) and (2.18). Therefore, two integrations by parts will yield

$$(2.28)\quad \int_t^\infty g(t_1)\exp\left\{-c_1t_1+\int_{t_0}^{t_1} f(t_2)dt_2\right\}dt_1$$

$$= -c_1^{-1}g(t)\exp\left\{-c_1t+\int_{t_0}^{t} f(t_2)dt_2\right\}$$

$$+c_1^{-2}\left[g'(t)+f(t)g(t)\right]\exp\left\{-c_1t+\int_{t_0}^{t} f(t_2)dt_2\right\}$$

$$+c_1^{-2}\int_t^\infty\left[g''+gf'+2g'f+gf^2\right]\exp\left\{-c_1t_1+\int_{t_0}^{t_1} f(t_2)dt_2\right\}dt_1.$$

Since $g(t)\exp\left\{-c_1t+\int_{t_0}^{t} f(t_2)dt_2\right\}$ is decreasing for $t \geq t_0$, the last term in (2.28) is bounded by

$$c_1^{-2}g(t)\exp\left\{-c_1t+\int_{t_0}^{t} f(t_2)dt_2\right\}\int_t^\infty\left(\left|\frac{g''}{g}\right|+|f'|+2\left|\frac{fg'}{g}\right|+f^2\right)dt_1$$

$$= g(t)\exp\left\{-c_1t+\int_{t_0}^{t} f(t_2)dt_2\right\}o(1).$$

Therefore (2.28) yields the stated conclusion.

Lemma 6. Suppose that μ is real and not zero, and that $g(t)$ satisfies these conditions:

$g(t)$ tends monotonically to zero as $t \to \infty$,

$g(t) \neq 0$ for $t \geq t_0$,

$g'(t) = o(g(t))$ as $t \to \infty$,

$$\int^\infty\left|\frac{g''(t)}{g(t)}\right|dt < \infty,\quad \int^\infty\left|\frac{g'(t)}{g(t)}\right|^2 dt < \infty.$$

Then as $t \longrightarrow \infty$

(2.29) $$\int_t^{\infty} e^{\mu t_1} g(t_1)dt_1 = -(\mu)^{-1} g(t)e^{\mu t}(1 + o(1)),$$

(2.30) $$\int_t^{\infty} e^{\mu t_1} g'(t_1)dt_1 = g(t)e^{\mu t}o(1),$$

and

(2.31) $$\int_t^{\infty} e^{\mu t_1} g(t_1)^2 dt_1 = -(\mu)^{-1} g(t)^2 e^{\mu t}(1 + o(1)).$$

Proof. Since $g(t) \longrightarrow 0$ and $g'(t) \longrightarrow 0$, two integrations by parts yield

$$\int_t^{\infty} e^{\mu t_1} g(t_1)dt_1 = -\frac{g(t)e^{\mu t}}{\mu} - \frac{g'(t)e^{\mu t}}{\mu^2} - \frac{1}{\mu^2}\int_t^{\infty} e^{\mu t_1} g''(t_1)dt_1.$$

Since

$$\left|\int_t^{\infty} e^{\mu t_1} g''(t_1)dt_1\right| \leq |g(t)| \int_t^{\infty} \left|\frac{g''(t_1)}{g(t_1)}\right| dt_1,$$

the relation in (2.29) is clear. Using a single integration by parts, we get

$$\int_t^{\infty} e^{\mu t_1} g'(t_1)dt_1 = -\frac{g'(t)e^{\mu t}}{\mu} - \frac{1}{\mu}\int_t^{\infty} e^{\mu t_1} g''(t_1)dt_1,$$

from which the relation in (2.30) follows. The relation in (2.31) is proved similarly.

Lemma 7. Suppose that $g(t)$ *satisfies the following conditions.*

$$g(t) > 0 \text{ for } t \geq t_0, \quad g(t) = o(1) \text{ as } t \to \infty, \tag{2.32}$$

$$g'(t) = o(g(t)^{3/2}), \quad g''(t) = o(g(t)^2) \text{ as } t \to \infty, \tag{2.33}$$

$$\int^{\infty} g(t)^{1/2} dt = \infty, \tag{2.34}$$

$$\int^{\infty} \frac{|g'(t)|^2}{g(t)^{5/2}} dt < \infty, \quad \int^{\infty} \frac{|g''(t)|}{g(t)^{3/2}} dt < \infty. \tag{2.35}$$

Given any real number n, *it then follows that*

$$\int_{t_0}^{t} g(t_1)^n \exp\left\{\int^{t_1} g(t_2)^{1/2} dt_2\right\} dt_1 \tag{2.36}$$

$$= (1 + o(1)) g(t)^{n-1/2} \exp\left\{\int^{t} g(t_1)^{1/2} dt_1\right\} \text{ as } t \to \infty.$$

Lemma 8. If $g(t)$ *satisfies the hypotheses in Lemma 7,*

$$\int_{t}^{\infty} g(t_1)^n \exp\left\{-\int^{t_1} g(t_2)^{1/2} dt_2\right\} dt_1 \tag{2.37}$$

$$= (1 + o(1)) g(t)^{n-1/2} \exp\left\{-\int^{t} g(t_1)^{1/2} dt_1\right\} \text{ as } t \to \infty.$$

The proof of Lemma 7 is similar to that of Lemma 3. After two integrations by parts, we have

$$(2.38)\qquad \int_{t_0}^{t} g(t_1)^n \exp\left\{\int^{t_1} g(t_2)^{1/2} dt_2\right\} dt_1$$

$$= \left\{g(t)^{n-1/2} - (n - \tfrac{1}{2}) g'(t) g(t)^{n-2}\right\} \exp\left\{\int^{t} g(t_1)^{1/2} dt_1\right\}$$

$$+ \text{constant} + (n - \tfrac{1}{2}) J(t_0, t),$$

where

$$J(t_0,t) = \int_{t_0}^{t} \left[(n-2) g'(t_1)^2 g(t_1)^{n-3} + g''(t_1) g(t_1)^{n-2}\right]$$

$$\cdot \exp\left\{\int^{t_1} g(t_2)^{1/2} dt_2\right\} dt_1 .$$

Since $g'(t) = o(g(t)^{3/2})$, the expression

$$g(t)^{n-1/2} \exp\left\{\int^{t} g(t_1)^{1/2} dt_1\right\}$$

$$= c \exp\left\{\int^{t} (n - \tfrac{1}{2}) \frac{g'(t_1)}{g(t_1)} dt_1 + \int^{t} g(t_1)^{1/2} dt_1\right\}$$

approaches $+\infty$ as $t \to +\infty$, and is therefore of higher order than the constant in the right member of (2.38). Furthermore,

$$g'(t) g(t)^{n-2} = o(g(t)^{n-1/2}).$$

Therefore, in order to complete the proof we need only show that

$$(2.39)\qquad J(t_0,t) = o(1) g(t)^{n-1/2} \exp\left\{\int^{t} g(t_1)^{1/2} dt_1\right\}.$$

We can suppose t_0 is so large that

$$g(t)^{n-1/2}\exp\left\{\int^t g(t_1)^{1/2}dt_1\right\}$$

is an increasing function of t, since changing t_0 affects only the constant in the right member of (2.38). Since

$$\int_{t_0}^{\infty} \left|g'(t_1)^2 g(t_1)^{-5/2} + g''(t_1)g(t_1)^{-3/2}\right|dt_1 < \infty,$$

by the hypothesis in (2.35), we therefore have

$$\begin{aligned}\left|J(t_0,\tfrac{t}{2})\right| &\leq c g(\tfrac{t}{2})^{n-1/2}\exp\left\{\int^{t/2} g(t_1)^{1/2}dt_1\right\}\\ &\leq c g(t)^{n-1/2}\exp\left\{\int^{t} g(t_1)^{1/2}dt_1\right\}J_1(t)\end{aligned}$$

where

$$\begin{aligned}J_1(t) &= \exp\left\{-\int_{t/2}^{t} g(t_1)^{1/2}dt_1 - (n-\tfrac{1}{2})\int_{t/2}^{t}\frac{g'(t_1)}{g(t_1)}dt_1\right\}\\ &\leq \exp\left\{-\tfrac{1}{2}\int_{t/2}^{t} g(t_1)^{1/2}dt_1\right\}.\end{aligned}$$

By the hypothesis in (2.33), given any $\varepsilon > 0$, there exists a $t_0 > 0$ such that

$$\frac{g'(t)}{g(t)^{3/2}} \geq -\varepsilon, \quad t \geq t_0.$$

Integrating this relation, we find that

$$g(t)^{-1/2} \leq \varepsilon t, \quad t \geq t_0',$$

and therefore that

$$\lim_{t\to\infty} \int_{t/2}^{t} g(t_1)^{1/2}dt_1 = \infty .$$

It follows that $J_1(t) = o(1)$, and thus that $J(t_0,t/2)$ is of the order indicated in the right member of (2.39). Furthermore,

$$|J(\tfrac{t}{2},t)| \leq g(t)^{n-1/2}\exp\left\{\int^{t} g(t_1)^{1/2}dt_1\right\} o(1).$$

This inequality establishes the relation in (2.39), and completes the proof of Lemma 7.

The proof of Lemma 8 is similar, and is omitted.

3. Principal Root Real and Simple

We shall begin our discussion of the linear equation

(3.1) $$u'(t) + (a_0 + a(t))u(t) + (b_0 + b(t))u(t - \omega) = 0,$$

in which

(3.2) $$a(t) \to 0 \text{ and } b(t) \to 0 \text{ as } t \to \infty,$$

by finding the asymptotic form of a solution corresponding to the principal characteristic root λ, which we shall assume to be <u>real</u> and <u>simple</u>. The case in which the principal root is multiple, and that in which there are complex principal roots, will be taken up in subsequent sections. Since λ is a simple root, it satisfies the relations

$$(3.3) \qquad \lambda + a_0 + b_0 e^{-\omega\lambda} = 0,$$

$$(3.4) \qquad 1 - b_0 \omega e^{-\omega\lambda} \neq 0.$$

The first step in establishing the existence of a solution associated with the root λ is, as in the case of ordinary differential equations, the conversion of the differential-difference equation into an integral equation. If we write the equation in (3.1) in the form

$$(3.5) \qquad u'(t) + a_0 u(t) + b_0 u(t - \omega) = - a(t)u(t) - b(t)u(t - \omega),$$

we see from the equation in (2.8) that every solution also satisfies the integral equation

$$(3.6) \qquad u(t) = u(t_0)k(t - \omega) - b_0 \int_{t_0-\omega}^{t_0} u(t_1)k(t - t_1 - \omega)dt_1$$

$$- \int_{t_0}^{t} \left[a(t_1)u(t_1) + b(t_1)u(t_1 - \omega)\right] k(t - t_1)dt_1,$$

$$t > t_0,$$

where t_0 is arbitrary. Here $k(t)$ is the sum of the residues of $e^{ts}h^{-1}(s)$ at the zeros of $h(s)$. Since the residue at $s = \lambda$ is

$$(3.7) \qquad \frac{e^{\lambda t}}{h'(\lambda)} = \frac{e^{\lambda t}}{1 - b_0 \omega e^{-\omega\lambda}} = c_1 e^{\lambda t},$$

we can write

(3.8) $$k(t) = c_1 e^{\lambda t} + k_1(t),$$

where $k_1(t)$ is the sum of the residues arising from zeros of $h(s)$ with real parts less than λ. Hence

(3.9) $$|k_1(t)| \leq ce^{kt}, \quad t \geq 0, \quad k < \lambda,$$

where c is a positive constant[1]. By taking $u(t) = 0$ over the initial interval $t_0 - \omega \leq t \leq t_0$, we find that there is one solution satisfying the equation

(3.10) $$u(t) = -\int_{t_0}^{t} \left[a(t_1)u(t_1) + b(t_1)u(t_1 - \omega)\right] k(t - t_1)dt_1, \quad t > t_0.$$

Moreover, since any constant times $e^{\lambda t}$ is a solution of the homogeneous equation

(3.11) $$u'(t) + a_0 u(t) + b_0 u(t - \omega) = 0,$$

there are solutions of the equation in (3.1) satisfying the equation

(3.12) $$u(t) = ce^{\lambda t} - \int_{t_0}^{t} \left[a(t_1)u(t_1) + b(t_1)u(t_1 - \omega)\right] k(t - t_1)dt_1, \quad t > t_0,$$

[1] Throughout this paper, the letter c will denote a quantity, not necessarily the same in any two appearances, which is independent of t. Such a symbol has been called a <u>generic symbol</u> for a constant. On the other hand, the letters $c_1, c_2, \ldots$, carrying numerical subscripts, denote specific constants, whose values remain unchanged throughout any <u>one</u> section of this paper.

for arbitrary c and t_0.

At this point we shall make the simplifying assumption that $b(t) = 0$. This assumption, which will be dropped in §4, enables us to present many of the principal features of our method, while avoiding certain difficulties which require a somewhat more complicated analysis. The integral equation now takes the form

$$(3.13)\qquad u(t) = ce^{\lambda t} - ce^{\lambda t}\int_{t_0}^{t} e^{-\lambda t_1} a(t_1)u(t_1)dt_1$$

$$-\int_{t_0}^{t} a(t_1)u(t_1)k_1(t - t_1)dt_1 .$$

It is particularly important to keep these integrals separate—the one containing the contribution of the principal root and the other the contribution of all other roots—as they will be found to have different orders of magnitude. If we make the assumption that

$$(3.14)\qquad \int^{\infty} |a(t)|\, dt < \infty ,$$

we can use (3.13) directly to show that $u(t)$ must be asymptotic to a constant times $e^{\lambda t}$. However, (3.14) is a more severe restriction than we wish to impose, and we therefore have to replace (3.13) by a more suitable equation. If we let

$$(3.15)\qquad p(t) = -\int_{t_0}^{t} a(t_1)u(t_1)k_1(t - t_1)dt_1 ,$$

and differentiate the equation in (3.13), we obtain

$$(3.16) \qquad u'(t) = \lambda\left[u(t) - p(t)\right] - c_1 a(t)u(t) + p'(t).$$

Letting

$$(3.17) \qquad w(t) = u(t) - p(t),$$

this can be put in the form

$$(3.18) \qquad w'(t) = \lambda w(t) - c_1 a(t)w(t) - c_1 a(t)p(t).$$

The form of this equation suggests that we adopt a technique useful in discussing the asymptotic behavior of solutions of the differential equation

$$(3.19) \qquad w'(t) = \left[\lambda - c_1 a(t)\right] w(t) + f(t),$$

in which $a(t)$ approaches zero as $t \longrightarrow \infty$. One of the most powerful such techniques is the use of a <u>Liouville transformation</u> of the independent variable, $s = s(t)$, where

$$(3.20) \qquad s(t) = \int_{t_0}^{t} \lambda(t_1)dt_1,$$

$$(3.21) \qquad \lambda(t) = \lambda - c_1 a(t).$$

This converts (3.18) into the form

$$(3.22) \qquad \frac{dw}{ds} = w(s) - \frac{c_1 a(t)p(t)}{\lambda(t)},$$

provided that $\lambda(t) \neq 0$ for $t \geq t_0$. Every solution of the

equation in (3.22) satisfies

$$(3.23) \qquad w(s) = ce^{s} - c_1 \int_{s_0}^{s} e^{s-s_1} \frac{a(t_1)p(t_1)}{\lambda(t_1)} ds_1,$$

where s_1 and t_1 are related by (3.20). If we now return to the original variables, we obtain

$$(3.24) \qquad u(t) = ce^{s(t)} + p(t) - c_1 e^{s(t)} \int_{t_0}^{t} e^{-s(t_1)} a(t_1)p(t_1)dt_1, \qquad t \geq t_0.$$

The assumption $\lambda(t) \neq 0$ for $t \geq t_0$ is actually unnecessary, since the equation in (3.24) can be obtained directly from the equation in (3.18) by use of an integrating factor, obviating the need to divide by $\lambda(t)$.

Taking $c = 1$, we have a solution of the equation in (3.1) which satisfies the integral equation

$$(3.25) \qquad u(t) = e^{s(t)} + p(t) - c_1 e^{s(t)} \int_{t_0}^{t} e^{-s(t_1)} a(t_1)p(t_1)dt_1, \qquad t > t_0.$$

We shall now show that the asymptotic behavior of $u(t)$ can be deduced from (3.25), provided $a(t)$ satisfies conditions which enable us to apply Lemma 2 or Lemma 3. We first establish that $u(t)e^{-s(t)}$ is bounded as $t \to \infty$. From (3.15) and (3.9) we have

$$|p(t)| \leq ce^{kt} \int_{t_0}^{t} |a(t_1)|\, |u(t_1)| e^{-kt_1} dt_1$$

and if we let

$$(3.26) \qquad m(t) = \max_{t_0 \le t_1 \le t} |u(t_1)e^{-s(t_1)}|,$$

we get

$$|p(t)| \le cm(t)e^{kt}\int_{t_0}^{t} |a(t_1)|e^{s(t_1)-kt_1}dt_1$$

or

$$(3.27) \qquad |p(t)| \le cm(t)e^{kt}\int_{t_0}^{t} |a(t_1)| \exp\left\{(\lambda - k)t_1 - c_1\int_{t_0}^{t_1} a(t_2)dt_2\right\}dt_1.$$

If $\int^{\infty} |a(t_1)|\, dt_1 < \infty$, Lemma 2 at once yields

$$(3.28) \qquad |p(t)| = m(t)e^{\lambda t}o(1) = m(t)e^{s(t)}o(1).$$

On the other hand, if the hypotheses of Lemma 3 are satisfied when $f(t) = -c_1a(t)$, $g(t) = |a(t)|$, then

$$(3.29) \qquad |p(t)| \le cm(t)e^{s(t)}|a(t)|.$$

From (3.28), we have

$$\left|\int_{t_0}^{t} e^{-s(t_1)}a(t_1)p(t_1)dt_1\right| \le om(t)\int_{t_0}^{\infty} |a(t_1)|\, dt_1$$

and from (3.29) we have

$$\left|\int_{t_0}^{t} e^{-s(t_1)} a(t_1)p(t_1)dt_1\right| \leq cm(t)\int_{t_0}^{\infty} a^2(t_1)dt_1.$$

In either case, then,

$$(3.30) \qquad \left|c_1\int_{t_0}^{t} e^{-s(t_1)} a(t_1)p(t_1)dt_1\right| \leq c_2 m(t),$$

where the constant c_2 is as small as desired if we choose t_0 large enough. Using these results in (3.25), we obtain

$$|u(t)e^{-s(t)}| \leq 1 + m(t)o(1) + c_2 m(t).$$

Therefore, by choosing t_0 sufficiently large, we can deduce

$$(3.31) \qquad |u(t)| \leq 2e^{s(t)}, \quad t \geq t_0.$$

By use of this inequality in (3.25), we can show that $u(t)e^{-s(t)}$ approaches a constant as $t \to \infty$. First of all, from (3.28) and (3.29) we get, respectively,

$$(3.32) \qquad |p(t)| \leq ce^{s(t)}o(1)$$

and

$$(3.33) \qquad |p(t)| \leq ce^{s(t)}|a(t)|.$$

In either case,

$$\int_{t_0}^{\infty} e^{-s(t_1)} a(t_1)p(t_1)dt_1$$

is absolutely convergent. Therefore, we can write (3.25) in

the form

$$(3.34) \qquad u(t) = c_3 e^{s(t)} + p(t) + c_1 e^{s(t)} \int_t^{\infty} e^{-s(t_1)} a(t_1) p(t_1) dt_1, \quad t \geq t_0,$$

where

$$c_3 = 1 - c_1 \int_{t_0}^{\infty} e^{-s(t_1)} a(t_1) p(t_1) dt_1 .$$

If t_0 is sufficiently large, $c_3 \neq 0$. It is clear from (3.34) that

$$(3.35) \qquad u(t) = c_3 e^{s(t)} (1 + o(1)) \quad \text{as} \quad t \to \infty .$$

In summary, we have proved the following theorem.

Theorem 1. Suppose that the principal characteristic root of $h(s) = s + a_0 + b_0 e^{-\omega s}$ lies at $s = \lambda$ and is real and simple. Let $a(t)$ satisfy one of the following two sets of hypotheses:

I $\int^{\infty} |a(t)| \, dt < \infty ;$

II $a(t) \to 0$ as $t \to \infty$,

$a(t) \neq 0$ for $t \geq t_0$,

$a'(t) = o(a(t))$ as $t \to \infty$,

$$\int^{\infty} a^2(t) dt < \infty , \int^{\infty} |a'(t)| \, dt < \infty , \int^{\infty} \left| \frac{a''(t)}{a(t)} \right| dt < \infty .$$

Then the equation

$$(3.36) \qquad u'(t) + (a_0 + a(t))u(t) + b_0 u(t-\omega) = 0$$

has a solution $u(t)$ of the form

$$(3.37) \qquad u(t) = ce^{s(t)}(1 + o(1))$$

$$= (1 + o(1))\exp\left\{\lambda t - c_1 \int_{t_0}^{t} a(t_1)dt_1\right\}$$

where $c_1 e^{\lambda t}$ is the residue of $e^{ts}h^{-1}(s)$ at $s = \lambda$,

$$(3.38) \qquad c_1 = (1 - b_0 \omega e^{-\omega\lambda})^{-1}.$$

4. Principal Root Real and Simple (continued)

The key steps in the above method were the representation of solutions by means of integral operators, as in (3.13), differentiation of this relation to yield an integro-differential equation, and the use of a Liouville transformation to yield an improved integral operator representation, as in (3.25). We wish to show now that essentially the same procedure can be used to discuss the equation

$$(4.1) \qquad u'(t) + (a_0 + a(t))u(t) + (b_0 + b(t))u(t-\omega) = 0,$$

where we no longer assume that $b(t) = 0$. As before, we assume that the principal root is real and simple and lies at $s = \lambda$. Instead of (3.13), we now have

$$(4.2) \qquad u(t) = ce^{\lambda t} - c_1 e^{\lambda t} \int_{t_0}^{t} e^{-\lambda t_1} \Big[a(t_1)u(t_1)$$

$$+ b(t_1)u(t_1 - \omega)\Big] dt_1 + p(t), \quad t > t_0,$$

where

$$(4.3) \qquad p(t) = -\int_{t_0}^{t} \Big[a(t_1)u(t_1) + b(t_1)u(t_1 - \omega)\Big] k_1(t - t_1)dt_1.$$

This time, when we differentiate (4.2), we obtain an integro-difference-differential equation

$$(4.4) \qquad u'(t) = \lambda\Big[u(t) - p(t)\Big] - c_1\Big[a(t)u(t)$$

$$+ b(t)a(t - \omega)\Big] + p'(t), \quad t > t_0,$$

rather than an integro-differential equation. The proper analogue of the substitution in (3.20) is not immediately obvious. This additional complication can be handled in the following way. It is not unreasonable to suppose that the solutions of (4.1) will have asymptotic behavior of the same general kind as solutions of (3.41), and in particular that $u(t) - u(t - \omega)e^{\lambda\omega}$ will be of lower order of magnitude than $u(t)$ itself. We therefore introduce the function

$$(4.5) \qquad v(t) = u(t) - u(t - \omega)e^{\omega\lambda}$$

and write the equation in (4.4) in the form

$$(4.6) \qquad u'(t) = \lambda[u(t) - p(t)] - c_1[a(t) + e^{-\omega\lambda}b(t)]u(t) + c_1e^{-\omega\lambda}b(t)v(t) + p'(t), \quad t > t_0.$$

We again let

$$(4.7) \qquad w(t) = u(t) - p(t),$$

so that

$$(4.8) \qquad w'(t) = \lambda w(t) - c_1[a(t) + e^{-\omega\lambda}b(t)]\, w(t) + c_1e^{-\omega\lambda}b(t)v(t) - c_1[a(t) + e^{-\omega\lambda}b(t)]p(t), \quad t > t_0.$$

We now make the Liouville transformation $s = s(t)$, where

$$(4.9) \qquad s(t) = \int_{t_0}^{t} \lambda(t_1)dt_1$$

and

$$(4.10) \qquad \lambda(t) = \lambda - c_1[a(t) + e^{-\omega\lambda}b(t)].$$

We obtain

$$(4.11) \qquad \frac{dw}{ds} = w + \frac{c_1e^{-\omega\lambda}b(t)v(t)}{\lambda(t)} - c_1\frac{a(t) + e^{-\omega\lambda}b(t)}{\lambda(t)}p(t),$$

provided that $\lambda(t) \neq 0$ for $t \geq t_0$. Just as in §3, this leads us to the equation

$$(4.12)\qquad u(t) = p(t) + e^{s(t)} - c_1 e^{s(t)} \int_{t_0}^{t} e^{-s(t_1)}\Big[a(t_1) + e^{-\omega\lambda} b(t_1)\Big] p(t_1)\,dt_1 + c_2 e^{s(t)} \int_{t_0}^{t} e^{-s(t_1)} b(t_1) v(t_1)\,dt_1, \quad t > t_0.$$

The assumption $\lambda(t) \neq 0$ for $t \geq t_0$ is actually unnecessary, since the equation in (4.12) can be obtained directly from the equation in (4.8) by use of an integrating factor without dividing by $\lambda(t)$.

Before we can estimate the magnitude of $u(t)$, we must obtain a similar representation of $v(t)$, for it is necessary to show that $v(t)$ is of lower order than $u(t)$. From (4.12), we readily deduce

$$(4.13)\qquad \begin{aligned} v(t) &= p(t) - p(t-\omega)e^{\omega\lambda} + q(t) \\ &\quad - c_1 q(t) \int_{t_0}^{t-\omega} e^{-s(t_1)} \Big[a(t_1) + e^{-\omega\lambda} b(t_1)\Big] p(t_1)\,dt_1 \\ &\quad - c_1 e^{s(t)} \int_{t-\omega}^{t} e^{-s(t_1)} \Big[a(t_1) + e^{-\omega\lambda} b(t_1)\Big] p(t_1)\,dt_1 \\ &\quad + c_2 q(t) \int_{t_0}^{t-\omega} e^{-s(t_1)} b(t_1) v(t_1)\,dt_1 \\ &\quad + c_2 e^{s(t)} \int_{t-\omega}^{t} e^{-s(t_1)} b(t_1) v(t_1)\,dt_1, \quad t > t_0 + \omega \end{aligned}$$

where

$$(4.14)\qquad q(t) = e^{s(t)} - e^{s(t-\omega)+\omega\lambda}$$

$$= e^{s(t)}\left\{1 - \exp\left(\omega\lambda - \int_{t-\omega}^{t} \lambda(t_1)dt_1\right)\right\}.$$

We now impose conditions on $a(t)$ and $b(t)$ of the type used in the preceding section. If

$$\text{I}\qquad \int^{\infty} |a(t)|\,dt < \infty,\quad \int^{\infty} |b(t)|\,dt < \infty,$$

the proof in §3 goes through without any change. We therefore concentrate on the case in which $a(t)$ and $b(t)$ satisfy the following conditions:

$$\text{II}\qquad a(t),\ b(t) \to 0 \quad \text{as} \quad t \to \infty;$$

$$a(t) \neq 0,\ b(t) \neq 0, \quad \text{for} \quad t \geq t_0;$$

$$a'(t) = o(a(t)),\ b'(t) = o(b(t)) \quad \text{as} \quad t \to \infty;$$

$$\int^{\infty} a^2(t)dt < \infty,\ \int^{\infty} |a'(t)|dt < \infty,\ \int^{\infty} \left|\frac{a''(t)}{a(t)}\right|dt < \infty;$$

$$\int^{\infty} b^2(t)dt < \infty,\ \int^{\infty} |b'(t)|dt < \infty,\ \int^{\infty} \left|\frac{b''(t)}{b(t)}\right|dt < \infty;$$

$$\int^{\infty} |a(t)b(t)|\,dt < \infty; \quad \text{and}$$

$$(4.15)\qquad \lim_{t\to\infty} \frac{a(t-\ell\omega)}{a(t)} = 1,\quad \lim_{t\to\infty} \frac{b(t-\ell\omega)}{b(t)} = 1,\quad 0 \leq \ell \leq 1.$$

It follows from (4.15) that

$$(4.16)\qquad \int_{t-\omega}^{t} a(t_1)dt_1 = O(a(t)),\quad \int_{t-\omega}^{t} b(t_1)dt_1 = O(b(t)),\quad t \to \infty.$$

It will be very convenient in what follows, whenever we are using the hypotheses II, to let the symbol $\varepsilon(t)$ denote a function, not necessarily the same in any two appearances, which is $O(|a(t+\omega)| + |b(t+\omega)|)$ as $t \to \infty$. Whenever we are using the hypotheses I, $\varepsilon(t)$ denotes a function which is simply $o(1)$. Thus we have, under either set of hypotheses,

$$(4.17) \qquad \int_{t-\omega}^{t} a(t_1)dt_1 = \varepsilon(t), \int_{t-\omega}^{t} b(t_1)dt_1 = \varepsilon(t),$$

and

$$(4.18) \qquad a(t) = \varepsilon(t), \quad a(t+\omega) = \varepsilon(t), \quad b(t) = \varepsilon(t),$$
$$b(t+\omega) = \varepsilon(t).$$

Note that under the hypotheses II, the product of two such functions is absolutely integrable over the infinite interval. In his notation, we have

$$(4.19) \qquad \lambda(t) = \lambda + \varepsilon(t)$$

and therefore, from (4.14),

$$(4.20) \qquad ;(t) = e^{s(t)}\varepsilon(t).$$

From the definition of $p(t)$ and the bound in (3.9), we get

$$|p(t)| \leq ce^{kt} \int_{t_0-\omega}^{t} (|a(t_1+\omega)|$$
$$+ |b(t_1+\omega)|)|u(t_1)|e^{-kt_1}dt_1.$$

We now let

$$(4.21)\qquad f(t) = a(t) + e^{-\omega\lambda}b(t)$$

and

$$(4.22)\qquad m(t) = \max_{t_0-\omega\leq t_1\leq t} \left|u(t_1)e^{-s(t_1)}\right|.$$

Then

$$(4.23)\qquad |p(t)| \leq cm(t)e^{kt}\int_{t_0-\omega}^{t} \varepsilon_1(t_1)\exp\left\{(\lambda - k)t_1 - c_1\int_{t_0}^{t_1} f(t_2)dt_2\right\}dt_1$$

where $\varepsilon_1(t)$ is a multiple of $|a(t_1 + \omega)| + |b(t_1 + \omega)|$. Under the hypotheses II, $a(t)$ and $b(t)$ are of constant sign for $t \geq t_0$. Hence the function $_1(t)$ is differentiable for $t \geq t_0 -$, and satisfies the hypotheses of Lemma 3. It therefore follows from Lemma 2 or Lemma 3 that

$$(4.24)\qquad p(t) = m(t)e^{s(t)}\varepsilon(t),\quad t \geq t_0.$$

From this last result, we also have

$$(4.25)\qquad \int_{t_0}^{t} e^{-s(t_1)}f(t_1)p(t_1)dt_1 \leq m(t)\int_{t_0}^{\infty} \varepsilon^2(t_1)dt_1,\quad t \geq t_0,$$

where $\varepsilon^2(t)$ is integrable. Furthermore, by (4.17),

$$(4.30)\qquad |v(t)e^{-s(t)}| \leq \varepsilon_2(t) + \varepsilon_2(t)\int_{t_0}^{t} e^{-s(t_1)}|b(t_1)||v(t_1)|dt_1$$

$$+\int_{t-\omega}^{t} e^{-s(t_1)}|b(t_1)|\,|v(t_1)|dt_1, \quad t \geq t_0 + \omega$$

where $\varepsilon_2(t)$ denotes a particular function of the type $\varepsilon(t)$.

Letting

$$(4.31)\qquad n(t) = \max_{t_0 \leq t_1 \leq t} \left|\frac{v(t_1)e^{-s(t_1)}}{\varepsilon_2(t_1)}\right|,$$

we then obtain

$$(4.32)\qquad |v(t)e^{-s(t)}| \leq \varepsilon_2(t) + \varepsilon_2(t)n(t)\int_{t_0}^{\infty} \varepsilon_2(t_1)|b(t_1)|dt_1$$

$$+ n(t)\varepsilon_2(t)\varepsilon(t),$$

from which

$$(4.33)\qquad \left|\frac{v(t)e^{-s(t)}}{\varepsilon_2(t)}\right| \leq 1 + n(t)\int_{t_0}^{\infty} \varepsilon_2(t_1)|b(t_1)|dt_1$$

$$+ n(t)\varepsilon(t), \quad t \geq t_0 + \omega$$

If t_0 is sufficiently large, it follows that $n(t) \leq c$ for $t \geq t_0$, and therefore

$$(4.34)\qquad |v(t)| \leq ce^{s(t)}\varepsilon_2(t), \quad t \geq t_0.$$

From (4.28) follows

$$(4.35) \qquad |u(t)| \leq ce^{s(t)}, \quad t \geq t_0 - \omega.$$

Now that the bounds in (4.34) and (4.35) have been established, the procedure in §3 can be used without essential change to show that

$$(4.36) \qquad u(t) = c_3 e^{s(t)}(1 + o(1)) \quad \text{as} \quad t \to \infty.$$

We have therefore proved the following theorem.

<u>Theorem 2.</u> <u>Suppose that the principal root of</u> $h(s) = s + a_0 + b_0 e^{-\omega s}$ <u>lies at</u> $s = \lambda$ <u>and is real and simple.</u> <u>Let</u>

$$(4.37) \qquad c_1 = 1/h'(\lambda) = (1 - b_0 \omega e^{-\omega\lambda})^{-1},$$

$$(4.10) \qquad \lambda(t) = \lambda - c_1 \left[a(t) + e^{-\omega\lambda} b(t)\right],$$

$$(4.9) \qquad s(t) = \int_{t_0}^{t} \lambda(t_1) dt_1.$$

<u>Suppose that</u> $a(t)$ <u>and</u> $b(t)$ <u>satisfy one of the following two sets of hypotheses:</u>

I $\quad \int^{\infty} |a(t)| dt < \infty, \quad \int^{\infty} |b(t)| dt < \infty.$

II $\quad a(t), b(t) \to 0$ <u>as</u> $t \to \infty$,

$a(t) \neq 0, \; b(t) \neq 0,$ <u>for</u> $t \geq t_0$,

$a'(t) = o(a(t)), \; b'(t) = o(b(t))$ <u>as</u> $t \to \infty$,

$$\int^{\infty} a^2(t)dt < \infty, \int^{\infty} |a'(t)|\, dt < \infty, \int^{\infty} \left|\frac{a''(t)}{a(t)}\right| dt < \infty,$$

$$\int^{\infty} b^2(t)dt < \infty, \int^{\infty} |b'(t)|\, dt < \infty, \int^{\infty} \left|\frac{b''(t)}{b(t)}\right| dt < \infty,$$

$$\int^{\infty} |a(t)b(t)|\, dt < \infty,$$

$$\lim_{t\to\infty} \frac{a(t-\ell\omega)}{a(t)} = 1, \quad \lim_{t\to\infty} \frac{b(t-\ell\omega)}{b(t)} = 1, \quad 0 \leq \ell \leq 1.$$

<u>Then the equation</u>

(4 1) $$u'(t) + (a_0 + a(t))u(t) + (b_0 + b(t))u(t-\omega) = 0$$

<u>has a solution</u> $u(t)$ <u>of the form</u>

(4.38) $$u(t) = e^{s(t)}(1 + o(1)) \quad \underline{\text{as}} \quad t \to \infty.$$

<u>5. Asymptotic Expansions</u>

In the two preceding sections, we have studied the asymptotic behavior of a solution of the differential-difference equation with "asymptotically constant" coefficients,

(5.1) $$u'(t) + (a_0 + a(t))u(t) + (b_0 + b(t))u(t-\omega) = 0,$$

(5.2) $$a(t) \to 0, \quad b(t) \to 0, \quad \text{as} \quad t \to \infty.$$

In this section, we shall suppose that $a(t)$ and $b(t)$ have asymptotic power series expansions, and shall show that the solution of (5.1) associated with the principal characteristic root has a corresponding asymptotic expansion. The theorem to be proved follows.

Theorem 3. *Suppose that the principal root of* $h(s) = s + a_0 + b_0 e^{-\omega s}$ *lies at* $s = \lambda$ *and is real and simple. Suppose that* $a(t)$ *and* $b(t)$ *have asymptotic power series expansions*

$$(5.3) \qquad a(t) \sim \sum_{n=1}^{\infty} a_n t^{-n}, \quad b(t) \sim \sum_{n=1}^{\infty} b_n t^{-n} \quad \text{as} \quad t \to \infty,$$

and that $a'(t)$, $b'(t)$, $a''(t)$, *and* $b''(t)$ *exist and have asymptotic power series expansions. Then there exists a solution* $u(t)$ *of the equation in* (5.1) *with an asymptotic expansion of the form*

$$(5.4) \qquad u(t) \sim e^{s(t)} \sum_{n=0}^{\infty} u_n t^{-n} \quad \text{as} \quad t \to \infty,$$

where each u_n *is a constant,* $u_0 \neq 0$, *and* $s(t)$ *is defined as in* §4. *Consequently,*

$$(5.5) \qquad u(t) \sim e^{\lambda t} t^r \sum_{n=0}^{\infty} u_n' t^{-n} \quad \text{as} \quad t \to \infty,$$

where each u_n' *is a constant,* $u_0' \neq 0$, *and where*

$$(5.6) \qquad r = -\frac{a_1 + b_1 e^{-\omega\lambda}}{1 - b_0 \omega e^{-\omega\lambda}} .$$

Proof. By hypothesis, $a'(t)$, $b'(t)$, $a''(t)$, and $b''(t)$ have asymptotic power series expansions, which must by (5.3) have the form

(5.7) $$a'(t) \sim -\frac{a_1}{t^2} - \frac{2a_2}{t^3} - \cdots, \quad b'(t) \sim -\frac{b_1}{t^2} - \frac{2b_2}{t^3} - \cdots,$$

$$a''(t) \sim \frac{2a_1}{t^3} + \frac{6a_2}{t^4} + \cdots, \quad b''(t) \sim \frac{2b_1}{t^3} + \frac{6b_2}{t^4} + \cdots .$$

It is clear that $a(t)$ and $b(t)$ satisfy the hypotheses of Theorem 2. Hence

(5.8) $$u(t) = e^{s(t)}(1 + o(1)) \quad \text{as} \quad t \to \infty,$$

where

(5.9) $$s(t) = \int_{t_0}^{t} \lambda(t_1)dt_1,$$

(5.10) $$\lambda(t) = \lambda - c_1\left[a(t) + e^{-\omega\lambda}b(t)\right].$$

If we write

(5.11) $$a(t) = \frac{a_1}{t} + a_2(t), \quad b(t) = \frac{b_1}{t} + b_2(t),$$

we have

$$s(t) = \lambda(t - t_0) - r\int_{t_0}^{t} \left[t_1^{-1} + a_2(t_1) + e^{-\omega\lambda}b_2(t_1)\right]dt_1.$$

Hence

(5.12) $$s(t) - \lambda t - r \log t \sim \sum_{n=0}^{\infty} s_n' t^{-n},$$

and

(5.13) $$e^{s(t)} \sim e^{\lambda t} t^{r} \sum_{n=0}^{\infty} s_n t^{-n},$$

where the s_n are constants and $s_0 \neq 0$. It is therefore clear that (5.4) and (5.5) are equivalent results.

We shall use an inductive method to establish the asymptotic expansion in (5.4). First of all, we shall show that

(5.14) $$u(t) = e^{s(t)}\left[u_0 + O(t^{-1})\right] \quad \text{as} \quad t \to \infty.$$

From (4.24) and (4.34), we have

(5.15) $$|p(t)| \leq c e^{s(t)} t^{-1},$$

(5.16) $$|v(t)| \leq c e^{s(t)} t^{-1}.$$

Therefore,

$$\int_{t_0}^{\infty} e^{-s(t_1)}\left[a(t_1) + e^{-\omega\lambda} b(t_1)\right] p(t_1)\,dt_1$$

and

$$\int_{t_0}^{\infty} e^{-s(t_1)} b(t_1) v(t_1)\,dt_1$$

are absolutely convergent, and the integral equation in (4.12) takes the form

$$(5.17)\qquad u(t) = c_3 e^{s(t)} + p(t) + c_1 e^{s(t)} \int_t^\infty e^{-s(t_1)} \left[a(t_1) + e^{-\omega\lambda} b(t_1)\right] p(t_1)dt_1 - c_2 e^{s(t)} \int_t^\infty e^{-s(t_1)} b(t_1)v(t_1)dt_1,$$

where

$$(5.18)\qquad c_3 = 1 - c_1 \int_{t_0}^\infty e^{-s(t_1)} \left[a + e^{-\omega\lambda} b\right] pdt_1 + c_2 \int_{t_0}^\infty e^{-s(t_1)} bvdt_1.$$

If t_0 is sufficiently large, $c_3 \neq 0$. Since the two integrals in (5.17) are easily seen to be $O(t^{-1})$, by (5.15) and (5.16), it is clear that

$$(5.19)\qquad u(t) = e^{s(t)} \left[c_2 + O(t^{-1})\right],$$

which proves (5.14).

We propose to use (5.14) to obtain more precise estimates of p and v than those in (5.15) and (5.16). Using (5.11) and (5.14), we obtain

$$(5.20)\qquad p(t) = -a_1u_0\int_{t_0}^{t} e^{s(t_1)}t_1^{-1}k_1(t-t_1)dt_1$$
$$-b_1u_0\int_{t_0}^{t} e^{s(t_1-\omega)}t_1^{-1}k_1(t-t_1)dt_1$$
$$+O\left(\int_{t_0}^{t} e^{s(t_1)}t_1^{-2}k_1(t-t_1)dt_1\right).$$

Since

$$(5.21)\qquad |k_1(t)| \leq ce^{kt}\quad (k<\lambda),$$

the last term in (5.20) is

$$(5.22)\qquad O\left(e^{kt}\int_{t_0}^{t} e^{s(t_1)-kt_1}t_1^{-2}dt_1\right) = O(e^{s(t)}t^{-2}),$$

by Lemma 3. To estimate the first integral in (5.20), we write

$$(5.23)\qquad \int_{t_0}^{t} e^{s(t_1)}t_1^{-1}k_1(t-t_1)dt_1 = J_1 + J_2,$$

where

$$(5.24)\qquad J_1 = \int_{t_0}^{t/2} e^{s(t_1)}t_1^{-1}k_1(t-t_1)dt_1,$$

$$(5.25)\qquad J_2 = \int_{t/2}^{t} e^{s(t_1)}t_1^{-1}k_1(t-t_1)dt_1.$$

Using (5.21), (5.13), and Lemma 3, we get

(5.26) $$|J_1| \leq t^{-1} \exp\left\{\frac{k}{2}t + s(\tfrac{t}{2})\right\} = O(e^{s(t)-\mathcal{E}(t)})$$

for some $\mathcal{E} > 0$. On the other hand,

(5.27) $$J_2 = \int_0^{t/2} \frac{e^{s(t-t_1)}}{t - t_1} k_1(t_1)dt_1$$

$$= \frac{e^{s(t)}}{t} \int_0^{t/2} e^{-\left[s(t)-s(t-t_1)\right]} \left(1 - \frac{t_1}{t}\right)^{-1} k_1(t_1)dt_1 .$$

Using (5.12), we find that

(5.28) $$s(t) - s(t - t_1) = \lambda t_1 + r \ln \frac{t}{t - t_1} - r_1(t,t_1),$$

where

(5.29) $$|r_1(t,t_1)| \leq \frac{t_1}{t(t - t_1)}, \quad 0 \leq t_1 < t.$$

Therefore

(5.30) $$\exp\left\{s(t - t_1) - s(t)\right\} = e^{-\lambda t_1}\left(1 - \frac{t_1}{t}\right)^r \exp r_1(t,t_1).$$

From this it is easy to see that if $0 \leq t_1 \leq t/2$, $\exp\left\{s(t - t_1) - s(t)\right\}$ is bounded as $t \rightarrow \infty$. In fact,

(5.31) $$e^{\lambda t_1}\left(1 - \frac{t_1}{t}\right)^{-1} \exp\left\{s(t - t_1) - s(t)\right\} = 1 + r_2(t,t_1)$$

where

(5.32) $$|r_2(t,t_1)| \leq \frac{ct_1}{t}, \quad 0 \leq t_1 \leq \frac{t}{2}.$$

Therefore

$$(5.33) \qquad J_2 = \frac{e^{s(t)}}{t} \int_0^{t/2} k_1(t_1) e^{-\lambda t_1} dt_1$$

$$+ \frac{e^{s(t)}}{t^2} O\left(\int_0^{t/2} k_1(t_1) t_1 e^{-\lambda t_1} dt_1\right).$$

Since $|k_1(t_1)| \leq c e^{+kt_1}$,

$$J_2 = \frac{e^{s(t)}}{t} \int_0^{\infty} k_1(t_1) e^{-\lambda t_1} dt_1 + O(e^{s(t)} t^{-2}),$$

$$(5.34) \qquad J_2 = c_4 e^{s(t)} t^{-1} + O(e^{s(t)} t^{-2}).$$

The equations in (5.26) and (5.34) provide an estimate of the first integral in (5.20), and the second can be treated in the same way. We therefore have

$$(5.35) \qquad p(t) = e^{s(t)} \left[\frac{p_1}{t} + O(t^{-2})\right]$$

where p_1 is a constant. It is clear from (4.14) and (5.12) that $q(t)$ has an asymptotic expansion of the form

$$(5.36) \qquad q(t) \sim e^{s(t)} \sum_{n=1}^{\infty} q_n t^{-n},$$

where the q_n are constants. The relations (5.11), (5.16), (5.35), and (5.36) can now be used in (4.13) to yield an improved estimate for $v(t)$. In fact,

$$q(t)\int_{t_0}^{t-\omega} e^{-s(t_1)}\left[a + e^{-\omega\lambda}b\right]p\,dt_1$$

$$= c_5 q(t) - q(t)\int_{t-\omega}^{\infty} e^{-s(t_1)}\left[a + e^{-\omega\lambda}b\right]p\,dt_1$$

$$= e^{s(t)}\left[c_5 t^{-1} + O(t^{-2})\right].$$

Also

$$e^{s(t)}\int_{t-\omega}^{t} e^{-s(t_1)}\left[a + e^{-\omega\lambda}b\right]p\,dt_1$$

$$= O(e^{s(t)}\int_{t-\omega}^{t} t_1^{-2}dt_1) = O(e^{s(t)}t^{-2}).$$

The other integrals in (4.13) are handled similarly, enabling us to deduce from (5.35) and (5.36) that

$$(5.37) \qquad v(t) = e^{s(t)}\left[\frac{v_0}{t} + O(t^{-2})\right],$$

where v_0 is a constant.

From the above discussion, it is seen that a knowledge of the relation (5.13) enables us to improve the bounds (5.15) and (5.16) on p and v so as to yield the estimates (5.35) and (5.37). On the other hand, (5.35) and (5.37) can now be inserted into (5.17) to improve the estimate of u. For example,

$$\int_t^\infty e^{-s(t_1)} b(t_1)v(t_1)dt_1 = \int_t^\infty \left[\frac{b_1 v_0}{t_1^2} + O(t_1^{-3})\right] dt_1$$

$$= \frac{b_1 v_0}{t} + O(t^{-2}).$$

Clearly we now get

$$(5.38) \qquad u(t) = e^{s(t)}\left[u_0 + \frac{u_1}{t} + O(t^{-2})\right] \quad \text{as} \quad t \to \infty.$$

It should now be evident that this process can be continued indefinitely. Each estimate of u enables us to obtain improved estimates of p and v,[1] and these enable us to obtain an improved estimate of u. Consequently $u(t)$ has a full asymptotic expansion of the form (5.4), or, equivalently, of the form (5.5).

Once we have established the existence of an asymptotic relation of the form in (5.5), the values of the coefficients u_n' and of the constant r can most easily be found for any particular equation (5.1) by substituting in (5.1) and equating coefficients of like powers of t.

6. Other Real, Simple Roots

We now wish to show that, associated with any real, simple characteristic root λ of

$$(6.1) \qquad u'(t) + (a_0 + a(t))u(t) + (b_0 + b(t))u(t - \omega) = 0,$$

[1] It is necessary to carry out the expansions in (5.28) and (5.31) to a greater number of terms in order to do this.

not necessarily the principal root, there is a solution $u(t)$ having the asymptotic form given by Theorem 2 or Theorem 3. As is true in the theory of ordinary differential equations, it is necessary in this situation somehow to suppress the influence of the roots having real parts greater than λ, which otherwise would dominate. We again start from the integral equation

$$u(t) = u(t_0)k(t-\omega) - b_0\int_{t_0-\omega}^{t_0} u(t_1)k(t-t_1-\omega)dt_1$$

$$-\int_{t_0}^{t}\left[a(t_1)u(t_1) + b(t_1)u(t_1-\omega)\right]k(t-t_1)dt_1,$$

$$t > t_0.$$

The solution which is zero for $t_0 - \omega \leq t \leq t_0$ therefore satisfies

$$u(t) = -\int_{t_0}^{t}\left[a(t_1)u(t_1) + b(t_1)u(t_1-\omega)\right]k(t-t_1)dt_1,$$

$$t > t_0,$$

and the solution which is equal to $ce^{\lambda t}$ for $t_0 - \omega \leq t \leq t_0$ satisfies

$$(6.2)\qquad u(t) = ce^{\lambda t} - \int_{t_0}^{t}\left[a(t_1)u(t_1)\right.$$

$$\left. + b(t_1)u(t_1-\omega)\right]k(t-t_1)dt_1.$$

This time we write

(6.3) $$k(t) = c_1 e^{\lambda t} + k_1(t) + k_2(t),$$

where $k_2(t)$ contains the finite number of residues of $e^{ts}h^{-1}(s)$ at zeros of $h(s)$ with real parts greater than λ . We have

(6.4) $$|k_1(t)| \leq ce^{kt}, \quad t \geq 0, \quad k < \lambda,$$

(6.5) $$|k_2(t)| \leq ce^{\ell t}, \quad t \geq 0, \quad \ell > \lambda.$$

Therefore

(6.6) $$\begin{aligned} u(t) = ce^{\lambda t} - c_1 e^{\lambda t} \int_{t_0}^{t} & [a(t_1)u(t_1) \\ & + b(t_1)u(t_1 - \omega)] e^{-\lambda t_1} dt_1 \\ - \int_{t_0}^{t} & [a(t_1)u(t_1) + b(t_1)u(t_1 - \omega)] k_1(t - t_1) dt_1 \\ - \int_{t_0}^{t} & [a(t_1)u(t_1) + b(t_1)u(t_1 - \omega)] k_2(t - t_1) dt_1 . \end{aligned}$$

Each term in $k_2(t)$ is a solution of the equation

(6.7) $$u'(t) + a_0 u(t) + b_0 u(t - \omega) = 0.$$

Hence

(6.8) $$\int_{t_0}^{\infty} [a(t_1)u(t_1) + b(t_1)u(t_1 - \omega)] k_2(t - t_1) dt_1$$

will also be a solution of the equation in (6.7), provided the integral converges. Assuming this to be true, addition of this integral to the right member of (6.6) produces a new integral equation

$$(6.9) \qquad u(t) = c_2 e^{\lambda t} - c_1 e^{\lambda t} \int_{t_0}^{t} \left[a(t_1)u(t_1) + b(t_1)u(t_1 - \omega)\right] e^{-\lambda t_1} dt_1 + p(t), \quad t \geq t_0,$$

where c_2 is arbitrary and

$$(6.10) \qquad p(t) = -\int_{t_0}^{t} \left[a(t_1)u(t_1) + b(t_1)u(t_1 - \omega)\right] k_1(t - t_1) dt_1 + \int_{t}^{\infty} \left[a(t_1)u(t_1) + b(t_1)u(t_1 - \omega)\right] k_2(t - t_1) dt_1 .$$

The solution of (6.9) satisfies

$$u(t) = c_2 e^{\lambda t} + \int_{t_0}^{\infty} \left[a(t_1)u(t_1) + b(t_1)u(t_1 - \omega)\right] k_2(t - t_1) dt_1 , \quad t_0 - \omega \leq t \leq t_0 .$$

If we differentiate the equation in (6.9), we obtain

$$u'(t) = \lambda\left[u(t) - p(t)\right] - c_1\left[a(t)u(t) + b(t)u(t - \omega)\right] + p'(t).$$

Letting

$$(6.11) \qquad w(t) = u(t) - p(t),$$

$$(6.12) \qquad v(t) = u(t) - u(t - \omega)e^{\omega\lambda},$$

we get

$$w'(t) = \lambda(t)w(t) + c_1 e^{-\omega\lambda} b(t)v(t) - c_1 f(t)p(t),$$

where

$$(6.13) \qquad f(t) = a(t) + e^{-\omega\lambda} b(t),$$

$$(6.14) \qquad \lambda(t) = \lambda - c_1 \left[a(t) + e^{-\omega\lambda} b(t)\right].$$

As in §4, we let $s = s(t)$, where

$$(6.15) \qquad s(t) = \int_{t_0}^{t} \lambda(t_1)dt_1,$$

and obtain the equations

$$(6.16a) \qquad u(t) = e^{\lambda(t-t_0)} + \int_{t_0}^{\infty} \left[a(t_1)u(t_1) + b(t_1)u(t_1 - \omega)\right] k_2(t - t_1)dt_1,$$

$$t_0 - \omega \leq t \leq t_0,$$

$$(6.16b) \quad u(t) = p(t) + e^{s(t)} - c_1 e^{s(t)} \int_{t_0}^{t} e^{-s(t_1)} f(t_1) p(t_1) dt_1$$

$$+ c_2 e^{s(t)} \int_{t_0}^{t} e^{-s(t_1)} b(t_1) v(t_1) dt_1, \quad t \geq t_0.$$

From (6.16), follows

$$(6.17) \quad v(t) = p(t) - p(t - \omega) e^{\omega\lambda} + q(t)$$

$$- c_1 q(t) \int_{t_0}^{t-\omega} e^{-s(t_1)} f(t_1) p(t_1) dt_1$$

$$- c_1 e^{s(t)} \int_{t-\omega}^{t} e^{-s(t_1)} f(t_1) p(t_1) dt_1$$

$$+ c_2 q(t) \int_{t_0}^{t-\omega} e^{-s(t_1)} b(t_1) v(t_1) dt_1$$

$$+ c_2 e^{s(t)} \int_{t-\omega}^{t} e^{-s(t_1)} b(t_1) v(t_1) dt_1, \quad t \geq t_0 + \omega,$$

where

$$(6.18) \quad q(t) = e^{s(t)} - e^{s(t-\omega)+\omega\lambda}.$$

It is, of course, not clear a priori that (6.1) has a solution for which the integral in (6.8) converges or which satisfies the integral equation in (6.9) and (6.16). We therefore have recourse to the method of successive approximations. By this method we shall establish the existence of a solution of (6.16) which is also a solution of (6.9). The equation in (6.9) has been so arranged that this solution will be found

to be of order $e^{\lambda t}$, roughly, and therefore the integral in (6.8) will be absolutely convergent and the solution of (6.9) will also be a solution of (6.1). We define successive approximations as follows:

$$(6.19) \qquad u_0(t) = e^{s(t)}, \quad t \geq t_0 - \omega.$$

$$(6.20a) \qquad u_{n+1}(t) = e^{\lambda(t-t_0)} + \int_{t_0}^{\infty} \left[a(t_1)u_n(t_1) + b(t_1)u_n(t_1 - \omega)\right] k_2(t - t_1)dt_1,$$

$$t_0 - \omega \leq t \leq t_0,$$

$$(6.20b) \qquad u_{n+1}(t) = p_n(t) + e^{s(t)} - c_1 e^{s(t)} \int_{t_0}^{t} e^{-s(t_1)} f(t_1)p_n(t_1)dt_1 + c_2 e^{s(t)} \int_{t_0}^{t} e^{-s(t_1)} b(t_1)v_n(t_1)dt_1, \quad t \geq t_0,$$

wherein we use the abbreviations

$$(6.21) \qquad p_n(t) = -\int_{t_0}^{t} \left[a(t_1)u_n(t_1) + b(t_1)u_n(t_1 - \omega)\right] k_1(t - t_1)dt_1 + \int_{t}^{\infty} \left[a(t_1)u_n(t_1) + b(t_1)u_n(t_1 - \omega)\right] k_2(t - t_1)dt_1,$$

$$t \geq t_0,$$

$$(6.22) \qquad v_n(t) = u_n(t) - u_n(t - \omega)e^{\omega\lambda}, \quad t \geq t_0.$$

Of course, it will be necessary to show that all the infinite integrals in these definitions converge for each n. From (6.17) it is clear that

$$(6.23)\qquad \begin{aligned} v_{n+1}(t) = {} & p_n(t) - p_n(t-\omega)e^{\omega\lambda} + q(t) \\ & - c_1 q(t)\int_{t_0}^{t-\omega} e^{-s(t_1)} f p_n \, dt_1 \\ & - c_1 e^{s(t)}\int_{t-\omega}^{t} e^{-s(t_1)} f p_n \, dt_1 \\ & + c_2 q(t)\int_{t_0}^{t-\omega} e^{-s(t_1)} b v_n \, dt_1 \\ & + c_2 e^{s(t)}\int_{t-\omega}^{t} e^{-s(t_1)} b v_n \, dt_1, \quad t \geq t_0 + \omega. \end{aligned}$$

We assume that $a(t)$ and $b(t)$ satisfy the hypotheses I or II of Theorem 2.

There is a constant $c_3 > 1$ such that

$$(6.24)\qquad e^{\lambda(t-t_0)} \leq c_3 e^{s(t)}, \quad t_0 - \omega \leq t \leq t_0.$$

We shall now prove that the integrals in (6.20a) and (6.21) converge for every n, and moreover that

$$(6.25)\qquad |u_n(t)| \leq 2c_3 e^{s(t)}, \quad t \geq t_0 - \omega, \quad n = 0,1,2,\dots,$$

$$(6.26)\qquad |v_n(t)| \leq 2c_3 \varepsilon_1(t) e^{s(t)}, \quad t \geq t_0, \quad n = 0,1,2,\dots,$$

where $\varepsilon_1(t)$ is a certain function of the type $\varepsilon(t)$. From

(6.19), the inequality in (6.25) is valid for $n = 0$. Also

$$(6.27) \qquad |v_0(t)| = |q(t)| = \mathcal{E}(t)e^{s(t)}.$$

Hence the inequality in (6.26) is valid for $n = 0$ if we choose $\varepsilon_2(t) \geq |q(t)e^{-s(t)}|$. We now proceed by induction. Suppose that the integrals in (6.20a) and (6.21) have been shown to converge for $n = 0,1,\dots,m-1$ and that the inequalities in (6.25) and (6.26) have been shown to be valid for $n = 0,1,\dots,m$. Under the hypotheses of Theorem 2, we can use Lemma 2 or Lemma 3 to show that

$$(6.28) \qquad \left|\int_{t_0}^{t} \left\{a(t_1)e^{s(t_1)} + b(t_1)e^{s(t_1-\omega)}\right\}k_1(t-t_1)dt_1\right|$$

$$\leq ce^{kt}\int_{t_0}^{t} \mathcal{E}(t_1)\exp\left\{(\lambda - k)t_1 - c_1\int_{t_0}^{t_1} f(t_2)dt_2\right\}dt_1$$

$$\leq \frac{1}{2}\, e^{s(t)}\varepsilon_2(t), \quad t \geq t_0.$$

Similarly, using Lemma 4 or Lemma 5,

$$(6.29) \qquad \left|\int_{t}^{\infty} \left\{a(t_1)e^{s(t_1)} + b(t_1)e^{s(t_1-\omega)}\right\}k_2(t-t_1)dt_1\right|$$

$$\leq \frac{1}{2}\, e^{s(t)}\varepsilon_2(t), \quad t \geq t_0.$$

It therefore follows from (6.25) with $n = m$ that the integrals in (6.20a) and (6.21) converge for $n = m$. Moreover, by using the bound $|u_m(t)| \leq 2c_3e^{s(t)}$ and the inequalities in (6.28) and (6.29), we find that

(6.30) $\quad |p_m(t)| \leq 2c_3 e^{s(t)} \varepsilon_2(t), \quad t \geq t_0.$

Also, from (6.20a) we get, using the inequality in (6.29),

$$|u_{m+1}(t)| \leq e^{\lambda(t-t_0)} + c_3 e^{s(t)} \varepsilon_2(t), \quad t_0 - \omega \leq t \leq t_0.$$

Using the inequality in (6.24), this yields

(6.31) $\quad |u_{m+1}(t)| \leq 2c_3 e^{s(t)}, \quad t_0 - \omega \leq t \leq t_0,$

since we can take t_0 sufficiently large that $\varepsilon_2(t_0) \leq 1$. From the inequality in (6.30), we deduce that

(6.32) $\quad \left| c_1 \int_{t_0}^{t} e^{-s(t_1)} f p_m dt_1 \right| \leq c_4, \quad t \geq t_0,$

where c_4 is as small as we please if t_0 is sufficiently large, and

(6.33) $\quad \left| c_1 \int_{t-\omega}^{t} e^{-s(t_1)} f p_m dt_1 \right| \leq \varepsilon_3(t), \quad t \geq t_0 + \omega.$

Using the inequalities in (6.30) and (6.32) and the inequality $|v_m(t)| \leq 2c_3 \varepsilon_1(t) e^{s(t)}$ in (6.20b), we get

$$|u_{m+1}(t)| \leq 2c_3 e^{s(t)} \varepsilon_2(t) + e^{s(t)} + c_4 e^{s(t)}$$

$$+ 2|c_2| c_3 e^{s(t)} \int_{t_0}^{\infty} |b(t_1)| \varepsilon_1(t_1) dt_1, \quad t \geq t_0.$$

If t_0 is so large that

$$(6.34) \qquad 2\varepsilon_2(t) + c_4c_3^{-1} + 2|c_2|\int_{t_0}^{\infty}|b(t_1)|\varepsilon_1(t_1)dt_1 < 1, \quad t \geq t_0,$$

it follows that

$$(6.35) \qquad |u_{m+1}(t)| \leq 2c_3e^{s(t)}, \quad t \geq t_0.$$

Finally, if we use the various inequalities already obtained, we get

$$\begin{aligned}|v_{m+1}(t)| \leq\ & 2c_3e^{s(t)}\varepsilon_2(t) + 2c_3e^{\omega\lambda}e^{s(t-\omega)}\varepsilon_2(t-\omega) \\ & + q(t)e^{s(t)} + c_4\varepsilon(t)e^{s(t)} + \varepsilon_3(t)e^{s(t)} \\ & + 2|c_2|c_3\varepsilon(t)e^{s(t)}\int_{t_0}^{\infty}|b(t_1)|\varepsilon_1(t_1)dt_1 \\ & + 2|c_2|c_3e^{s(t)}\int_{t-\omega}^{t}|b(t_1)|\varepsilon_1(t_1)dt_1, \quad t \geq t_0 + \omega\end{aligned}$$

and

$$\begin{aligned}|v_{m+1}(t)| &\leq |u_{m+1}(t)| + e^{\omega\lambda}|u_{m+1}(t-\omega)| \\ &\leq 4c_3e^{s(t)}, \quad t_0 \leq t \leq t_0 + \omega.\end{aligned}$$

By choice of $\varepsilon_1(t)$ and t_0, and use of (4.17), we deduce that

$$(6.36) \qquad |v_{m+1}(t)| \leq 2c_3\varepsilon_1(t)e^{s(t)}, \quad t \geq t_0.$$

This completes the inductive proof of the inequalities in (6.25) and (6.26).

Next, we prove that the sequence $\{u_n(t)\}$ converges as $n \to \infty$, uniformly in any finite interval in which $t \geq t_0 - \omega$. In order to do this, we shall show that

$$(6.37) \qquad |u_{n+1}(t) - u_n(t)| \leq 2^{-n+2} c_3 e^{s(t)}, \quad t \geq t_0 - \omega,$$

$$n = 0,1,2,\ldots,$$

and

$$(6.38) \qquad |v_{n+1}(t) - v_n(t)| \leq 2^{-n+2} c_3 \epsilon_1(t) e^{s(t)}, \quad t \geq t_0,$$

$$n = 0,1,2,\ldots .$$

Since the proof is by induction, and very similar to that just given, we shall omit it.

From (6.37) it is clear that the series

$$(6.39) \qquad \sum_{n=0}^{\infty} \{u_{n+1}(t) - u_n(t)\}$$

is absolutely and uniformly convergent for t in any finite interval in which $t \geq t_0 - \omega$. Let

$$(6.40) \qquad u(t) = \lim_{n \to \infty} u_n(t), \quad t \geq t_0 - \omega,$$

$$(6.41) \qquad v(t) = \lim_{n \to \infty} v_n(t), \quad t \geq t_0.$$

From the inequalities in (6.25) and (6.26), we have

$$(6.42) \qquad |u(t)| \leq 2c_3 e^{s(t)}, \quad t \geq t_0 - \omega,$$

$$(6.43)\qquad |v(t)| \leq 2c_3\varepsilon_1(t)e^{s(t)}, \quad t \geq t_0.$$

With the aid of these bounds, it is easily seen that the integral

$$(6.8)\qquad \int_{t_0}^{\infty} \left[a(t_1)u(t_1) + b(t_1)u(t_1 - \omega)\right] k_2(t - t_1)dt_1$$

is absolutely convergent. We can therefore define $p(t)$ as in (6.10). The uniform convergence of the sequence $\{u_n(t)\}$ to $u(t)$ enables us to deduce that $u(t)$ satisfies the equations in (6.16). Thus $u(t)$ is continuous for $t \geq t_0 - \omega$ and satisfies the equation in (6.9) for $t \geq t_0$. Since the integral in (6.8) is absolutely convergent, $u(t)$ is also a solution of the differential-difference equation in (6.1).

By using the bounds in (6.42) and (6.43), we can in the usual way deduce the asymptotic form of $u(t)$ from the integral equation in (6.16). We write

$$(6.44)\qquad u(t) = c_4e^{s(t)} + p(t) + c_1e^{s(t)}\int_t^{\infty} e^{-s(t_1)}\left[a + e^{-\omega\lambda}b\right]pdt_1$$
$$- c_2e^{s(t)}\int_t^{\infty} e^{-s(t_1)}bvdt_1,$$

where

$$(6.45)\qquad c_4 = 1 - c_1\int_{t_0}^{\infty} e^{-s(t_1)}\left[a + e^{-\omega\lambda}b\right]pdt_1$$
$$+ c_2\int_{t_0}^{\infty} e^{-s(t_1)}bvdt_1,$$

and obtain

$$(6.46)\qquad u(t) = c_4 e^{s(t)}(1 + o(1)) \quad \text{as} \quad t \to \infty .$$

Furthermore, if $a(t)$ and $b(t)$ have asymptotic power series expansions,

$$(6.47)\qquad a(t) \sim \sum_{n=1}^{\infty} a_n t^{-n}, \quad b(t) \sim \sum_{n=1}^{\infty} b_n t^{-n},$$

and if $a'(t)$, $b'(t)$, $a''(t)$, and $b''(t)$ exist and have asymptotic power series expansions, then the procedure of §5 shows that $u(t)$ has an asymptotic expansion. In fact, the equation in (6.16b) is identical in form with the equation in (4.12), and the only difference in the discussion is caused by the different form of $p(t)$ as given by (6.10). It is only necessary to add to the previous arguments an analysis of the integral

$$J = \int_t^{\infty} \left[a(t_1)u(t_1) + b(t_1)u(t_1 - \omega)\right] k_2(t - t_1)dt_1 .$$

For example, given that

$$u(t) = e^{s(t)}\left[u_0 + O(t^{-1})\right],$$

we have

$$J = a_1 u_0 \int_t^{\infty} e^{s(t_1)} t_1^{-1} k_2(t - t_1)dt_1$$

$$+ b_1 u_0 \int_t^{\infty} e^{s(t_1 - \omega)} t_1^{-1} k_2(t - t_1)dt_1$$

$$+ O(\int_t^{\infty} e^{s(t_1)} t_1^{-2} k_2(t - t_1)dt_1).$$

The last term here is easily seen to be $O(e^{s(t)}/t^2)$. To estimate the first integral, we write

$$\int_t^{\infty} e^{s(t_1)} t_1^{-1} k_2(t - t_1)dt_1 = \int_{-\infty}^{0} \frac{e^{s(t-t_1)}}{t - t_1} k_2(t_1)dt_1$$

$$= \frac{e^{s(t)}}{t} \int_{-\infty}^{0} e^{-[s(t)-s(t-t_1)]} (1 - \frac{t_1}{t})^{-1} k_2(t_1)dt_1.$$

As in §5, this yields

$$\int_t^{\infty} e^{s(t_1)} t_1^{-1} k_2(t - t_1)dt_1$$

$$= \frac{e^{s(t)}}{t} \int_{-\infty}^{0} e^{-\lambda t_2} k_2(t_2)dt_2 + O(\frac{e^{s(t)}}{t^2}).$$

The infinite integral is convergent because $|k_2(t)| \leq ce^{\ell t}$, $\ell > \lambda$. It is thus clear that the entire discussion in §5 can be carried through just as before.

The results of this section can be summarized by saying that <u>Theorems 2 and 3 remain valid if λ is any real, simple characteristic root, not necessarily the principal root</u>.

7. Complex Roots of Multiplicity One

We shall now show how to find the asymptotic form of a solution associated with any simple root, real or complex, of the equation

$$(7.1)\qquad u'(t) + (a_0 + a(t))u(t) + (b_0 + b(t))u(t - \omega) = 0.$$

Let the root in question be λ, and let

$$(7.2)\qquad c_1 = 1/h'(\lambda),$$

$$(7.3)\qquad \lambda(t) = \lambda - c_1\left[a(t) + b(t)e^{-\omega\lambda}\right],$$

$$(7.4)\qquad s(t) = \int_{t_0}^{t} \lambda(t_1)dt_1.$$

The results of §4 and §6 suggest that there will be a solution of the form

$$u(t) \sim e^{s(t)}.$$

However, there is an additional difficulty in showing this directly by the method used in §6, because Lemmas 3 and 5 cannot be applied to an integral of the form

$$e^{(\mathrm{Re}\ \lambda + i\mu)t}\int \left\{a(t_1)u(t_1) + b(t_1)u(t_1 - \omega)\right\} e^{-(\mathrm{Re}\ \lambda + i\mu)t_1}dt_1,$$

which can now appear in the expression for $p(t)$ since there are other roots with real part equal to the real part of λ. The simplest way to get around this difficulty is to make the substitution

$$(7.5)\qquad u(t) = e^{s(t)}x(t).$$

The equation in (7.1) then takes the form

$$(7.6)\qquad x'(t) + [a_0 + a(t) + \lambda(t)]x(t)$$
$$+ [b_0 + b(t)]\, e^{s(t-\omega)-s(t)}x(t-\omega) = 0.$$

Letting

$$(7.7)\qquad f(t) = a(t) + b(t)e^{-\omega\lambda},$$

we have

$$(7.8)\qquad s(t-\omega) - s(t) = -\omega\lambda + c_1\omega f(t) - \tfrac{1}{2}c_1\omega^2 f'(t) + O(f''(t)),$$

and

$$(7.9)\qquad e^{s(t-\omega)-s(t)} = e^{-\omega\lambda}\left[1 + c_1\omega f(t) - \tfrac{1}{2}c_1\omega^2 f'(t) + O(f''(t))\right].$$

The equation in (7.6) therefore can be written

$$(7.10)\qquad x'(t) + [a_0 + a(t) + \lambda(t)]x(t) + e^{-\omega\lambda}[b_0 + b(t)]\Big[1 + c_1\omega f(t)$$
$$- \tfrac{1}{2}c_1\omega^2 f'(t) + O(f''(t))\Big]x(t-\omega) = 0.$$

The characteristic equation for (7.10) is

$$(7.11)\qquad s + \lambda + a_0 + b_0e^{-\omega(s+\lambda)} = 0,$$

from which we see that the transformation in (7.5) has trans-

lated all roots by the amount $-\lambda$. In particular, the root in which we are interested now lies at $s = 0$. Furthermore, since $c_1(1 - b_0\omega e^{-\omega\lambda}) = 1$,

$$(7.12)\qquad a_0 + a(t) + \lambda(t) + e^{-\omega\lambda}[b_0 + b(t)][1 + c_1\omega f(t) - \tfrac{1}{2}c_1\omega^2 f'(t) + O(f''(t))]$$

$$= e^{-\omega\lambda}c_1\omega b(t)f(t) - \tfrac{1}{2}e^{-\omega\lambda}b_0c_1\omega^2 f'(t) + O(bf' + f'').$$

We shall denote a function of the type in (7.12) by the symbol $\varepsilon^2(t)$, assuming the hypotheses of §4 and §6 are satisfied.

Let us therefore consider an equation of the form

$$(7.13)\qquad u'(t) + (a_0 + a(t))u(t) + (b_0 + b(t))u(t - \omega) = 0,$$

having a root at $s = 0$ and complex coefficients, such that $a_0 + b_0 = 0$ and such that $a(t) + b(t) = \varepsilon^2(t)$, as in (7.12). We shall show that there is a solution which is asymptotically constant. The kernel function associated with (7.13) now has the form

$$(7.14)\qquad k(t) = c_1 + c_2e^{i\mu t} + k_1(t) + k_2(t),$$

where

$$(7.15)\qquad c_1 = \frac{1}{h'(0)} = \frac{1}{1 - b_0\omega},$$

and

$$(7.16) \qquad |k_1(t)| \leq ce^{kt}, \quad k < 0,$$

$$(7.17) \qquad |k_2(t)| \leq ce^{\ell t}, \quad \ell > 0,$$

and where $c_2 e^{i\mu t}$ represents the contribution of the other root lying on the imaginary axis. (For higher order equations, there can be more than one other, but for an equation of the form in (7.13), there are at most two roots on any vertical line, both of them simple.) As before, we have solutions which satisfy

$$(7.18) \qquad u(t) = c + ce^{i\mu t} - \int_{t_0}^{t} [a(t_1)u(t_1) + b(t_1)u(t_1 - \omega)] k(t - t_1)dt_1, \quad t \geq t_0.$$

Assuming that the integrals

$$(7.19) \qquad \int_{t_0}^{\infty} |a(t_1)u(t_1) + b(t_1)u(t_1 - \omega)| dt_1$$

and

$$(7.20) \qquad \int_{t_0}^{\infty} |a(t_1)u(t_1) + b(t_1)u(t_1 - \omega)| \, |k_2(t - t_1)| dt_1$$

are convergent, we can replace the integral equation in (7.18) by

$$(7.21)\qquad u(t) = 1 - \int_{t_0}^{t} \left[a(t_1)u(t_1) + b(t_1)u(t_1 - \omega)\right] k_1(t - t_1)dt_1$$

$$+ \int_{t}^{\infty} \left[a(t_1)u(t_1) + b(t_1)u(t_1 - \omega)\right] k_2(t - t_1)dt_1$$

$$+ c_1 \int_{t}^{\infty} \left[a(t_1)u(t_1) + b(t_1)u(t_1 - \omega)\right] dt_1$$

$$+ c_2 e^{1/\mu t} \int_{t}^{\infty} \left[a(t_1)u(t_1) + b(t_1)u(t_1 - \omega)\right] e^{-1/\mu t_1} dt_1,$$

$$t \geq t_0.$$

The corresponding equation for $t_0 - \omega \leq t \leq t_0$ is

$$(7.22)\qquad u(t) = 1 + \int_{t_0}^{\infty} \left[a(t_1)u(t_1) + b(t_1)u(t_1 - \omega)\right] k_2(t - t_1)dt_1$$

$$+ c_1 \int_{t_0}^{\infty} \left[a(t_1)u(t_1) + b(t_1)u(t_1 - \omega)\right] dt_1$$

$$+ c_2 e^{1/\mu t} \int_{t_0}^{\infty} \left[a(t_1)u(t_1) + b(t_1)u(t_1 - \omega)\right] e^{-1/\mu t_1} dt_1.$$

If we put

$$(7.23)\qquad v(t) = u(t) - u(t - \omega),$$

we can write

$$(7.24)\qquad a(t)u(t) + b(t)u(t - \omega) = \left[a(t) + b(t)\right] u(t) - b(t)v(t),$$

$$t \geq t_0.$$

We now define iterates $u_n(t)$, $n = 0,1,2,\ldots,$ as follows:

(7.25) $$u_0(t) = 1, \quad t \geq t_0 - \omega,$$

(7.26) $$u_{n+1}(t) = 1 + \int_{t_0}^{\infty} \left[a(t_1)u_n(t_1) + b(t_1)u_n(t_1-\omega)\right] k_2(t-t_1)dt_1$$

$$+ c_1 \int_{t_0}^{\infty} \left[a(t_1)u_n(t_1) + b(t_1)u_n(t_1-\omega)\right] dt_1$$

$$+ c_2 e^{i\mu t} \int_{t_0}^{\infty} \left[a(t_1)u_n(t_1) + b(t_1)u_n(t_1-\omega)\right] e^{-i\mu t_1} dt_1,$$

$$t_0 - \omega \leq t \leq t_0,$$

(7.27) $$u_{n+1}(t) = 1 - \int_{t_0}^{t} \left[a(t_1)u_n(t_1) + b(t_1)u_n(t_1-\omega)\right] k_1(t-t_1)dt_1$$

$$+ \int_{t}^{\infty} \left[a(t_1)u_n(t_1) + b(t_1)u_n(t_1-\omega)\right] k_2(t-t_1)dt_1$$

$$+ c_1 \int_{t}^{\infty} \left[a(t_1)u_n(t_1) + b(t_1)u_n(t_1-\omega)\right] dt_1$$

$$+ c_2 e^{i\mu t} \int_{t}^{\infty} \left[a(t_1)u_n(t_1) + b(t_1)u_n(t_1-\omega)\right] e^{-i\mu t_1} dt_1,$$

$$t \geq t_0.$$

We shall now show that there is a function $\epsilon_1(t)$ such that

(7.28) $$|u_n(t)| \leq 2, \quad t \geq t_0 - \omega, \quad n = 0,1,2,\ldots,$$

$$(7.29) \qquad |v_n(t)| \leq 2\varepsilon_1(t), \quad t \geq t_0, \quad n = 0,1,2,\ldots,$$

and such that the infinite integrals in (7.26) and (7.27) converge. From the definition in (7.25), it is clear that the inequalities in (7.28) and (7.29) are valid for $n = 0$. We now proceed by induction. Suppose that the integrals in (7.26) and (7.27) have been shown to be convergent for $n = 0,1,\ldots,m-1$, and that the inequalities in (7.28) and (7.29) have been shown to be valid for $n = 0,1,\ldots,m$. Then from (7.24) we find that

$$(7.30) \qquad |a(t)u_m(t) + b(t)u_m(t-\omega)| \leq 2|a(t) + b(t)| + 2|b(t)|\varepsilon_1(t),$$

$$(7.31) \qquad |a(t)u_m(t) + b(t)u_m(t-\omega)| \leq \varepsilon^2(t), \quad t \geq t_0,$$

where $\varepsilon^2(t)$ denotes the product of two functions of the type $\varepsilon(t)$. It follows that the integrals are convergent for $n = m$. Moreover, by Lemma 2 or 3,

$$(7.32) \qquad \int_{t_0}^{t} |a(t_1)u_m(t_1) + b(t_1)u_m(t_1-\omega)|\,|k_1(t-t_1)|\,dt_1$$

$$\leq e^{kt}\int_{t_0}^{t} \varepsilon^2(t_1)e^{-kt_1}dt_1 \leq \varepsilon(t), \quad (k < 0)$$

and by Lemma 4 or 5,

$$(7.33)\qquad \int_t^{\infty} |a(t_1)u_m(t_1) + b(t_1)u_m(t_1 - \omega)|\, |k_2(t - t_1)|\, dt_1$$

$$\leq e^{\ell t}\int_t^{\infty} \varepsilon^2(t_1)e^{-\ell t_1}dt_1 \leq \varepsilon(t) \quad (\ell > 0).$$

Also, by Lemma 6,

$$(7.34)\qquad e^{\mu t}\int_t^{\infty} \left[a(t_1)u_m(t_1) + b(t_1)u_m(t_1 - \omega)\right] e^{-\mu t_1}dt_1$$

$$= \varepsilon(t)o(1).$$

Finally,

$$(7.35)\qquad \int_t^{\infty} |a(t_1)u_m(t_1) + b(t_1)u_m(t_1 - \omega)|\, dt_1$$

$$= \int_t^{\infty} \varepsilon^2(t_1)dt_1 = o(1).$$

It is therefore clear from the definitions in (7.26) and (7.27) that

$$(7.36)\qquad |u_{m+1}(t)| \leq 2, \quad t \geq t_0 - \omega$$

provided t_0 is sufficiently large. Moreover, if we use the abbreviation

$$w_m(t) = a(t)u_m(t) + b(t)u_m(t - \omega),$$

we have

$$(7.37) \qquad v_{m+1}(t) = -\int_{t_0}^{t} w_m(t_1)k_1(t - t_1)dt_1$$

$$+\int_{t_0}^{t-\omega} w_m(t_1)k_1(t - \omega - t_1)dt_1$$

$$+\int_{t}^{\infty} w_m(t_1)k_2(t - t_1)dt_1$$

$$-\int_{t-\omega}^{\infty} w_m(t_1)k_2(t - \omega - t_1)dt_1$$

$$+ c_2 e^{i\mu t}\int_{t}^{\infty} w_m(t_1)e^{-i\mu t_1}dt_1$$

$$- c_2 e^{i\mu(t-\omega)}\int_{t-\omega}^{\infty} w_m(t_1)e^{-i\mu t_1}dt_1$$

$$- c_1\int_{t-\omega}^{t} w_m(t_1)dt_1, \quad t \geq t_0 - \omega.$$

The first six integrals in the right member of (7.37) are all $\mathcal{E}(t)$, by the inequalities in (7.32), (7.33), and (7.34). The last is

$$- c_1\int_{t-\omega}^{t} w_m(t_1)dt_1 = - c_1 w_m(t - \bar{\omega}) = o(\mathcal{E}_1(t)).$$

$$(0 < \bar{\omega} < \omega).$$

It follows that if $\mathcal{E}_1(t)$ is properly chosen and t_0 is sufficiently large,

$$|v_{m+1}(t)| \leq 2\mathcal{E}_1(t), \quad t \geq t_0 + \omega.$$

Since

$$|v_{m+1}(t)| \leq |u_{m+1}(t)| + |u_{m+1}(t - \omega)| \leq 4,$$

$$t_0 \leq t \leq t_0 - \omega,$$

we have

(7.38) $$|v_{m+1}(t)| \leq 2(\varepsilon_1(t)), \quad t \geq t_0.$$

This completes the inductive proof of the relations in (7.28) and (7.29).

The rest of the proof is very similar to that in §6, and we shall omit the details. The sequence $\{u_n(t)\}$ converges uniformly to a limit function $u(t)$ for which

$$|u(t)| \leq 2, \quad t \geq t_0 - \omega,$$

$$|v(t)| \leq 2\varepsilon_1(t), \quad t \geq t_0.$$

This limit function satisfies the integral equations in (7.21) and (7.22). Since

$$a(t)u(t) + b(t)u(t - \omega) = \left[a(t) + b(t)\right]u(t)$$

$$- b(t)v(t) = \varepsilon(t)^2,$$

the infinite integrals in (7.19) and (7.20) are convergent, and therefore $u(t)$ is a solution of the equation in (7.13). From the equation in (7.21), it is evident that

$u(t) = 1 + O(1)$. Moreover, if

$$a(t) \sim \sum_{n=1}^{\infty} a_n t^{-n}, \quad b(t) \sim \sum_{n=1}^{\infty} b_n t^{-n}, \quad a_1 + b_1 = 0,$$

we can in the usual way show from (7.21) that

$$u(t) \sim \sum_{n=0}^{\infty} u_n t^{-n}.$$

Taking account of the preliminary transformation in (7.5), we see that we have proved the following theorems, which include Theorems 2 and 3 as special cases.

<u>Theorem 4. Let</u> λ <u>be any simple root of the characteristic function</u> $h(s) = s + a_0 + b_0 e^{-\omega s}$. <u>Let</u>

$$c_1 = 1/h'(\lambda) = (1 - b_0 \omega e^{-\omega\lambda})^{-1},$$

$$\lambda(t) = \lambda - c_1 \left[a(t) + e^{-\omega\lambda} b(t)\right],$$

$$s(t) = \int_{t_0}^{t} \lambda(t_1) dt_1.$$

<u>Suppose that</u> $a(t)$ <u>and</u> $b(t)$ <u>satisfy one of the following two sets of hypotheses</u>:

I $\quad \int^{\infty} |a(t)|\, dt < \infty, \quad \int^{\infty} |b(t)|\, dt < \infty;$

II $\quad a(t)$ <u>and</u> $b(t)$ <u>tend monotonically to zero as</u> $t \to \infty$,

$$a'(t) = o(a(t)), \quad b'(t) = o(b(t)) \text{ as } t \to \infty,$$

$$\int^{\infty} a^2(t)dt < \infty, \quad \int^{\infty} |a'(t)|\,dt < \infty,$$

$$\int^{\infty} \left|\frac{a''(t)}{a(t)}\right| dt < \infty,$$

$$\int^{\infty} b^2(t)dt < \infty, \quad \int^{\infty} |b'(t)|\,dt < \infty,$$

$$\int^{\infty} \left|\frac{b''(t)}{b(t)}\right| dt < \infty,$$

$$\int^{\infty} |a(t)b(t)|\,dt < \infty,$$

$$\lim_{t\to\infty} \frac{a(t - \ell\omega)}{a(t)} = 1, \quad \lim_{t\to\infty} \frac{b(t - \ell\omega)}{b(t)} = 1, \quad 0 \leq \ell \leq 1.$$

Then the equation

$$u'(t) + (a_0 + a(t))u(t) + (b_0 + b(t))u(t - \omega) = 0$$

has a solution of the form

$$u(t) = e^{s(t)}(1 + o(1)) \quad \text{as} \quad t \to \infty.$$

Theorem 5. Let λ be any simple root of the characteristic function $h(s) = s + a_0 + b_0 e^{-\omega s}$. Suppose that $a(t)$ and $b(t)$ have asymptotic power series expansions

$$a(t) \sim \sum_{n=1}^{\infty} a_n t^{-n}, \quad b(t) \sim \sum_{n=1}^{\infty} b_n t^{-n},$$

and that $a'(t)$, $b'(t)$, $a''(t)$, and $b''(t)$ exist and have asymptotic power series expansions. Then there exists a solution $u(t)$ of the equation

$$u'(t) + (a_0 + a(t))u(t) + (b_0 + b(t))u(t - \omega) = 0$$

with an asymptotic expansion of the form

$$u(t) \sim e^{\lambda t} t^r \sum_{n=0}^{\infty} u_n t^{-n}$$

where each u_n is a constant, $u_0 \neq 0$, and

$$r = -\frac{a_1 + b_1 e^{-\omega\lambda}}{1 - b_0 \omega e^{-\omega\lambda}} .$$

8. Multiple Characteristic Roots

We now wish to examine the nature of a solution corresponding to a multiple root of the equation

$$(8.1) \qquad u'(t) + (a_0 + a(t))u(t) + (b_0 + b(t))u(t - \omega) = 0,$$

or more generally, the nature of a solution of a vector-matrix equation of the form

$$(8.2) \qquad x'(t) = \sum_{i=0}^{m} A_i(t)x(t - \omega_i),$$

where the corresponding characteristic equation has a multiple root. It turns out, as might be expected from a knowledge of the corresponding situation for ordinary differential equations, that the expansion of a solution corresponding to a multiple root is not as simple as that of a solution corresponding to a simple root. An idea of what to expect can be obtained by consulting the paper of Yates,

who showed that a solution of the equation with linear coefficients,

$$(8.3) \qquad x'(t) = \sum_{i=0}^{m} (tA_i + B_i)x(t - \omega_i),$$

corresponding to a <u>double</u> root λ, takes one of the forms

$$(8.4) \qquad x(t) = ct^r e^{\alpha t^{1/2}}(1 + o(1)),$$

$$x(t) = ct^r(1 + o(1)) + c,$$

$$x(t) = c \log t(1 + o(1)) + c,$$

where r and α are constants depending on the equation in (8.3), and where the c's are constant vectors.

Results of a similar type occur in the theory of ordinary differential equations. For example, it is known[1] that the equation

$$(8.5) \qquad u''(t) + a(t)u(t) = 0$$

in which

$$(8.6) \qquad a(t) \sim \sum_{n=1}^{\infty} a_n t^{-n} \quad (a_1 \neq 0),$$

has two solutions of the form

$$(8.7) \qquad u(t) \sim t^{1/4} e^{\alpha t^{1/2}} \sum_{n=0}^{\infty} u_n t^{-n/2}.$$

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[1] Cf. Erdelyi, <u>Asymptotic Expansions</u>, Dover Publications, Inc., 1956, p. 63.

The asymptotic series in (8.7) contains powers of $t^{-1/2}$ rather than powers of t.

We begin by showing how the method of the <u>Liouville transformation</u> can be used in ascertaining the asymptotic form of solutions of ordinary differential equations with multiple characteristic roots, since we propose to show in later sections that the same technique is of great usefulness in discussing differential-difference equations. Let us therefore consider a second order differential equation, with asymptotically constant coefficients, for which the characteristic equation has a real double root. Such an equation has the form

$$(8.8) \qquad u''(t) - 2[a + a_1(t)]u'(t) + [a^2 + a_2(t)]u(t) = 0,$$

where we suppose that $a_1(t)$ and $a_2(t)$ are real and

$$(8.9) \qquad a_1(t) \to 0 \quad a_2(t) \to 0,$$

as $t \to \infty$. The characteristic equation, $s^2 - 2as + a^2 = 0$, has a double root at $s = a$. As a first step, we shall translate this double root to the point $s = 0$. To do this, we shall use the following lemma. These and the subsequent lemmas may be found in [1], Chapter 6.

<u>Lemma 9. The substitution</u>

$$(8.10) \qquad u = v \exp\left\{-\frac{1}{2}\int^t p(t)dt\right\}$$

transforms

(8.11) $$u'' + p(t)u' + q(t)u = 0$$

into

(8.12) $$v'' + \left[q(t) - \tfrac{1}{2}p'(t) - \tfrac{1}{4}p(t)^2\right] v = 0.$$

This lemma can be verified directly. When applied to the equation in (8.8), it results in an equation of the form

(8.13) $$v''(t) - b(t)v(t) = 0,$$

where

(8.14) $$b(t) = -a_2(t) + 2aa_1(t) - a_1'(t) + a_1(t)^2.$$

The function $b(t)$ will approach zero as $t \to \infty$, but will not in general be integrable over the infinite interval. We therefore make a Liouville transformation, as in the following lemma.

Lemma 10. The change of variable

(8.15) $$s = \int_{t_0}^{t} a(t_1)dt_1, \quad a(t) > 0 \text{ for } t \geq t_0,$$

transforms

(8.16) $$u''(t) \pm a(t)^2 u(t) = 0$$

into

(8.17) $$\frac{d^2u}{ds^2} + \frac{a'(t)}{a(t)^2}\frac{du}{ds} \pm u = 0.$$

This lemma is also easy to verify. Let us assume that $b(t) \neq 0$ for $t \geq t_0$. Then if we apply Lemma 10 to the equation in (8.13) we obtain

(8.18) $$\frac{d^2v}{ds^2} + \frac{c'(t)}{c(t)^2}\frac{dv}{ds} \mp v = 0$$

where

(8.19) $$c(t) = |b(t)|^{1/2} = \{\pm b(t)\}^{1/2}.$$

Once again we use Lemma 9 to eliminate the middle term in (8.18). The result is the equation

(8.20) $$\frac{d^2w}{ds^2} + [\mp 1 + f(s)]\,w = 0,$$

where

(8.21) $$f(s) = -\frac{1}{2}\frac{d}{ds}\left(\frac{c'(t)}{c(t)^2}\right) - \frac{1}{4}\left(\frac{c'(t)}{c(t)^2}\right)^2.$$

In many cases, the function $f(s)$ will be integrable over the infinite interval. Moreover, the two characteristic roots of the equation in (8.20) have been <u>separated</u>. Assuming that $f(s)$ is integrable, we next transform the equation in (8.20) into an integral equation, using the following lemma.

<u>Lemma 11. Let</u> u_1 <u>and</u> u_2 <u>be linear independent</u>

<u>solutions of the equation</u>

$$(8.22)\qquad u''(t) + a(t)u(t) = 0$$

<u>for which the Wronskian</u>

$$(8.23)\qquad W(u_1,u_2) = \begin{vmatrix} u_1 & u_2 \\ u_1' & u_2' \end{vmatrix} = 1$$

<u>for all</u> t. <u>Then every solution of the inhomogeneous equation</u>

$$(8.24)\qquad u''(t) + a(t)u(t) = f(t)$$

<u>satisfies an equation of the form</u>

$$(8.25)\qquad u(t) = c_1u_1(t) + c_2u_2(t) + \int_{t_0}^{t} \left[u_1(t)u_2(t_1) - u_1(t_1)u_2(t)\right]f(t_1)dt_1.$$

Writing the equation in (8.20) in the form

$$(8.26)\qquad \frac{d^2w}{ds^2} \mp w = - f(s)w,$$

and applying Lemma 11, we find that every solution satisfies an integral equation

$$(8.27)\qquad w(s) = c_1e^{is} + c_2e^{-is} + \frac{1}{2i}\int_{s_0}^{s} \left[e^{i(s-s_1)} - e^{-i(s-s_1)}\right]f(s_1)w(s_1)ds_1,$$

or

$$(8.28) \qquad w(s) = c_1 e^{s} + c_2 e^{-s} + \frac{1}{2}\int_{s_0}^{s} \left[e^{s-s_1} - e^{-(s-s_1)}\right] f(s_1) w(s_1) ds_1,$$

depending on the sign in (8.26). If $f(s)$ is absolutely integrable over the infinite interval, and if $s \longrightarrow \infty$ as $t \rightarrow \infty$, it is easy to show from these equations by a now familiar argument, cf. [1], that there are two solutions

$$(8.29) \qquad w_1(s) = e^{is}(1 + o(1)), \quad w_2(s) = e^{-is}(1 + o(1)),$$

or

$$(8.30) \qquad w_1(s) = e^{s}(1 + o(1)), \quad w_2(s) = e^{-s}(1 + o(1)),$$

as the case may be. From these results, it is easy to write down the asymptotic form of the solutions of the equation in (8.8).

We should like to illustrate these remarks by considering the case in which the coefficient function $b(t)$ in (8.13) has an asymptotic power series,

$$(8.31) \qquad b(t) \sim \sum_{n=1}^{\infty} b_n t^{-n}.$$

It is necessary to divide the discussion into three cases, according as $b_1 = b_2 = 0$, $b_1 = 0$ and $b_2 \neq 0$ or $b_1 \neq 0$. In the last case, the Liouville transformation has the form

$$s \sim 2|b_1|^{1/2} t^{1/2} \sum_{n=0}^{\infty} s_n t^{-n} \qquad (s_0 = 1)$$

and

$$f(s) = \frac{5}{16} \frac{|b'(t)|^2}{|b(t)|^3} \mp \frac{b''(t)}{4|b(t)|^2} .$$

It is readily seen that $f(s) = O(t^{-1}) = O(s^{-2})$. Therefore the above method shows that there are solutions of the form

$$w(s) \sim e^{\pm is} \sum_{n=0}^{\infty} c_n s^{-n} \qquad (b_1 > 0)$$

or

$$w(s) \sim e^{\pm s} \sum_{n=0}^{\infty} c_n s^{-n} \qquad (b_1 < 0).$$

Since $v(t) = w(s)|b(t)|^{-1/4}$, we obtain

$$(8.32) \qquad v(t) \sim t^{1/4} e^{\pm 2(-b_1 t)^{1/2}} \sum_{n=0}^{\infty} u_n t^{-n/2} \qquad (b_1 \neq 0).$$

Incidentally, once the existence of an asymptotic relation of the form

$$v(t) \sim t^{\alpha} e^{\beta t^{1/2}} \left[1 + \sum_{n=1}^{\infty} u_n t^{-n/2} \right]$$

has been established, the values of the coefficients u_n and the parameters α and β can most easily be determined by substituting in the differential equation in (8.13) and equating coefficients.

If $b_1 = 0$ and $b_2 \neq 0$, the Liouville transformation has the form

$$s - |b_2|^{1/2} \log t \sim \sum_{n=0}^{\infty} s_n t^{-n}.$$

In this case,

$$f(s) = -\frac{1}{4|b_2|} + f_2(s)$$

where

$$f_2(s) \sim \sum_{n=1}^{\infty} c'_n t^{-n} \sim \sum_{n=1}^{\infty} c_n e^{-ns/|b_2|^{1/2}}.$$

The transformed equation has the form

$$\frac{d^2w}{ds^2} \mp c_3^2 w = -f_2(s)w \tag{8.33}$$

where

$$c_3 = \left|1 + \frac{1}{4b_2}\right|^{1/2}.$$

The upper and lower signs in (8.33) are to be used according as $b_2 > -1/4$ or $b_2 < -1/4$. It follows that there are solutions of the form

$$w(s) = e^{\pm c_3 s} \sum_{n=0}^{\infty} c_n e^{-ns/|b_2|^{1/2}}, \quad b_2 > -\frac{1}{4}, \tag{8.34}$$

or

$$(8.35) \qquad w(s) = e^{\pm ic_3 s} \sum_{n=0}^{\infty} c_n e^{-ns/|b_2|^{1/2}}, \quad b_2 < -\frac{1}{4}.$$

The form of $v(t)$ is readily deduced from these relations. On the other hand, if $b_2 = -1/4$, then $c_3 = 0$, and from the equation in (8.33) we get

$$(8.36) \qquad w(s) = c_1 + c_2 s + \int_{s_0}^{s} (s - s_1) f_2(s_1) w(s_1) ds_1.$$

It can be shown that there are solutions of the form

$$w(s) = 1 + o(1) \quad \text{and} \quad w(s) = s(1 + o(1))$$

and therefore

$$(8.37) \qquad v(t) = t^{1/2}(1 + o(1)) \quad \text{and} \quad v(t) = t^{1/2} \log t(1 + o(1)).$$

Further terms in the asymptotic expansion can be found by closer examination of the equation in (8.36), using the known form of $f_2(s)$.

Finally, if $b_1 = 0$, $b_2 = 0$, the Liouville transformation is not needed, and should not be used inasmuch as the integral in (8.15) will not approach ∞ as $t \to \infty$. In this case, we work directly with the equation in (8.13). Applying Lemma 11, we obtain

$$(8.38) \qquad v(t) = c_1 + c_2 t + \int_{t_0}^{t} (t - t_1) b(t_1) v(t_1) dt_1.$$

Since

(8.39) $\int^{\infty} t|b(t)|\,dt$

is convergent, it is not hard to deduce from the equation in (8.38) that there are two solutions v_1 and v_2 having the forms

(8.40) $v_1 = 1 + o(1), \quad v_2 = t(1 + o(1)).$

The estimates in (8.40) can be carried out to as many terms as desired. (Some of the terms contain logarithms, in general.)

The method sketched here is applicable not only when the coefficient in (8.13) has an asymptotic power series expansion, but also when it is an arbitrary power of t, $\log t$, or e^t, or a combination of these. It may sometimes happen that $f(s)$ in the equation in (8.20) is not integrable, but that one or more repetitions of the transformation employed will yield an equation of the type in (8.20) in which $f(s)$ is integrable.

9. The Liouville Transformation in General

The extension of the Liouville transformation to linear systems of higher order is the following. In place of an n-th order equation, consider a vector-matrix system

(9.1) $\frac{dx}{dt} = A(t)x.$

Let $\lambda_1(t), \lambda_2(t), \ldots, \lambda_N(t)$ be the characteristic roots

of $A(t)$ assumed distinct for $t \geq t_0$. Let, under this assumption, $T(t)$ be a matrix which reduces $A(t)$ to diagonal form, i.e.,

$$(9.2) \qquad T^{-1}(t)A(t)T(t) = \begin{pmatrix} \lambda_1(t) & & & 0 \\ & \lambda_2(t) & & \\ & & \ddots & \\ 0 & & & \lambda_N(t) \end{pmatrix}.$$

Make the change of variable

$$(9.3) \qquad x = T(t)y.$$

Then (9.1) becomes

$$(9.4) \qquad \frac{dy}{dt} = \left[T^{-1}(t)A(t)T(t) - T'(t)\right]y.$$

One may either use this form or make the further changes of independent variable

$$(9.5) \qquad s = \int^t \lambda_1(t_1)dt_1,$$

dependent upon the solution that is being examined.

This method has been used by Cesari, [4], and Levinson, [6], to study the stability and asymptotic behavior of solutions of linear systems. The infinite-dimensional character of differential-difference equations forces us to use a different approach.

10. Multiple Roots of Differential-difference Equations: Principal Root

We shall now show how to obtain the asymptotic form of a solution, corresponding to a multiple root, λ, of a differential-difference equation of the form

$$(10.1) \qquad u'(t) + (a_0 + a(t))u(t) + (b_0 + b(t))u(t - \omega) = 0.$$

In this section, we shall suppose that λ is the principal root, and later we shall consider an arbitrary multiple root λ. The characteristic function for the equation in (10.1) is

$$(10.2) \qquad h(s) = s + a_0 + b_0 e^{-\omega s}.$$

Since

$$(10.3) \qquad h'(s) = 1 - b_0 \omega e^{-\omega s},$$

there is at most one multiple root, and if there is one, it is a real double root given by $\lambda = - a_0 - \omega^{-1}$. The technique sketched in §8 for differential equations suggests that we should begin by translating the root to the origin. This can be accomplished by the substitution

$$(10.4) \qquad u(t) = e^{\lambda t} u_1(t).$$

We may as well suppose, then, that the equation in (10.1) has a double root at $s = 0$, in which case

(10.5) $h(0) = a_0 + b_0 = 0,$

(10.6) $h'(0) = 1 - b_0\omega = 1 + a_0\omega = 0.$

After a short calculation, we find that the residue of $e^{ts}h^{-1}(s)$ at $s = 0$ has the form $c_1 + c_2 t$, where

(10.7) $c_1 = \dfrac{2}{3b_0\omega} = \dfrac{2}{3}, \quad c_2 = \dfrac{2}{b_0\omega^2} = \dfrac{2}{\omega}.$

Following the procedure in §3, we find that there are solutions of the equation in (10.1) which satisfy the integral equation

(10.8) $$u(t) = c + c't - c_1\int_{t_0}^{t} \left[a(t_1)u(t_1) + b(t_1)u(t_1 - \omega)\right] dt_1$$

$$- c_2\int_{t_0}^{t} (t - t_1)\left[a(t_1)u(t_1) + b(t_1)u(t_1 - \omega)\right]dt_1$$

$$+ p(t), \quad t \geq t_0,$$

where

(10.9) $$p(t) = -\int_{t_0}^{t} \left[a(t_1)u(t_1) + b(t_1)u(t_1 - \omega)\right] k_1(t - t_1)dt_1,$$

$$t \geq t_0.$$

The constants c and c' are arbitrary. Since all roots except $s = 0$ lie to the left of the imaginary axis,

(10.10) $\quad |k_1(t)| \leq ce^{kt}, \quad k < 0, \quad t \geq 0.$

As in our previous work, we extract a differential equation from (10.8) by differentiation. This time, two successive differentiations are required. We obtain

$$(10.11) \quad \begin{aligned} u''(t) - p''(t) = &- c_1a'(t)u(t) - c_1b'(t)u(t-\omega) \\ &- c_1a(t)u'(t) - c_1b(t)u'(t-\omega) \\ &- c_2a(t)u(t) - c_2b(t)u(t-\omega). \end{aligned}$$

In §4, at the corresponding point in our discussion, we put

$$(10.12) \quad v(t) = u(t) - u(t-\omega)$$

and replaced $u(t-\omega)$ by $u(t) - v(t)$. The success of this technique was contingent on the fact that $v(t)$ was of lower order of magnitude than $u(t)$ itself. If this unadorned technique is tried in the equation in (10.11), one finds that the proofs cannot be carried through in all cases. The reason for this is that the ratio $v(t)/u(t)$ may not be of sufficiently small order as $t \to \infty$. In order to handle all the cases in which we are interested, primarily those in which $a(t)$ and $b(t)$ have asymptotic power series expansions, we therefore find it essential to use a more refined analysis. In the last term in the right member in (10.11), we write

$$(10.13) \qquad u(t-\omega) = u(t) - \omega u'(t) + u_1(t).$$

The point of this is that $b(t)u_1(t)/u(t)$ will, in general, be of sufficiently small order, whereas $b(t)u'(t)/u(t)$ will not. It is not necessary, in the cases in which we are interested, to use a similar expansion of the terms $b'(t)u(t-\omega)$ and $b(t)u'(t-\omega)$. If we now use (10.12), we obtain the equation

$$\begin{aligned}(10.14) \qquad u''(t) - p''(t) = &- c_1a'(t)u(t) - c_1b'(t)u(t) \\ &+ c_1b'(t)v(t) - c_1a(t)u'(t) \\ &- c_1b(t)u'(t) + c_1b(t)v'(t) - c_2a(t)u(t) \\ &- c_2b(t)u(t) + c_2\omega b(t)u'(t) - c_2b(t)u_1(t).\end{aligned}$$

We let

$$(10.15) \qquad w(t) = u(t) - p(t),$$

and obtain

$$\begin{aligned}(10.16) \qquad w''(t) &+ \left[c_1a(t) + c_1b(t) - c_2\omega b(t)\right] \\ &+ \left[c_1a'(t) + c_1b'(t) + c_2a(t) + c_2b(t)\right]w(t) \\ &= p_1(t) + v_1(t),\end{aligned}$$

where

$$(10.17)\qquad p_1(t) = -c_1a'(t)p(t) - c_1b'(t)p(t) - c_1a(t)p'(t)$$

$$- c_1b(t)p'(t) - c_2a(t)p(t) - c_2b(t)p(t) + c_2\omega b(t)p'(t),$$

$$(10.18)\qquad v_1(t) = c_1b'(t)v(t) + c_1b(t)v'(t) - c_2b(t)u_1(t).$$

We now use Lemma 9 of §8 to eliminate the term containing $w'(t)$ in (10.16). The substitution

$$(10.19)\qquad w(t) = x(t)r(t),$$

where

$$(10.20)\qquad r(t) = \exp\left[-\frac{1}{2}\int^t \left\{c_1a(t_1) + c_1b(t_1) - c_2\omega b(t_1)\right\} dt_1\right]$$

leads to the equation

$$(10.21)\qquad x''(t) - g(t)x(t) = \frac{p_1(t) + v_1(t)}{r(t)}$$

where

$$(10.22)\qquad g(t) = -c_2\left[a(t) + b(t)\right] - \frac{1}{2}\left[c_1a'(t) + c_1b'(t) + c_2\, b'(t)\right]$$

$$+ \frac{1}{4}\left[c_1a(t) + c_1b(t) - c_2\omega b(t)\right]^2.$$

The equation in (10.19) corresponds to the differential equation in (8.13), which we discussed in detail in case

$$b(t) \sim \sum_{n=1}^{\infty} b_n t^{-n}.$$

We found it necessary to divide the discussion into three cases, according as $b_1 = b_2 = 0$, $b_1 = 0$ and $b_2 \neq 0$, or $b_1 \neq 0$. In the first case, we worked directly with the equation in (8.13), but in the other two cases we employed a Liouville transformation. The equation in (10.19) will be handled in a similar way, but under general hypotheses similar to those used in §4. In this section, we shall state the general hypotheses required and the conclusions reached, and we shall also discuss the application of these general results to equations whose coefficients have asymptotic power series. In §11, we shall prove the general theorems. In §12, we shall outline the extension to non-principal roots and to roots of higher multiplicity.

In our general theorems we shall deal with functions $g(t)$ satisfying one or another of the following four sets of hypotheses:

I

(10.23) $\quad t^2 g(t) = o(1),$

(10.24) $\quad g'(t) = o(g(t)),$

(10.25) $\quad \int^{\infty} t|g(t)|dt < \infty.$

II

(10.26) $\quad g(t) = c_0 t^{-2}(1 + O(t^{-1})).$

III

(10.27) $g(t) = o(1)$,

(10.28) $g(t) \neq 0$ for $t \geq t_0$,

(10.29) $g'(t) = o(g(t)^{3/2})$, $g''(t) = o(g(t)^2)$,

(10.30) $\int^{\infty} |g(t)|^{1/2} dt = \infty$, $\int^{\infty} |g(t)|\, dt < \infty$,

(10.31) $\int^{\infty} \frac{|g'(t)|^2}{|g(t)|^{5/2}} dt < \infty$, $\int^{\infty} \frac{|g''(t)|}{|g(t)|^{3/2}} dt < \infty$.

IV

(10.27) $g(t) = o(1)$,

(10.28) $g(t) \neq 0$ for $t \geq t_0$,

(10.32) $g'(t) = O(g(t)^2)$, $g''(t) = O(g(t)^3)$,

(10.33) $\int^{\infty} |g(t)|dt = \infty$, $\int^{\infty} |g(t)|^{3/2} dt < \infty$.

In every case we also ask that

(10.34) $\lim\limits_{t \to \infty} \frac{g(t - \omega \ell)}{g(t)} = 1$, $\lim\limits_{t \to \infty} \frac{g'(t - \omega \ell)}{g'(t)} = 1$,

for every number ℓ, $0 \leq \ell \leq 1$.

If $g(t)$ has the form

$$g(t) \sim t^{-\alpha} \sum_{n=0}^{\infty} g_n t^{-n}, \quad g_0 \neq 0,$$

$g(t)$ satisfies these hypotheses for the following values of α:

I $\alpha > 2$

II $\alpha = 2$

III $1 < \alpha < 2$

IV $\frac{2}{3} < \alpha \leq 1$.

We intend to prove the following theorem.

Theorem 6. Suppose that the principal root of $h(s) = s + a_0 + b_0 e^{-\omega s}$ lies at $s = 0$ and is a double root. Consider the equation

(10.1) $$u'(t) + (a_0 + a(t))u(t) + (b_0 + b(t))u(t - \omega) = 0$$

in which $a(t)$ and $b(t)$ are twice continuously differentiable. Let $r(t)$ and $g(t)$ be defined as in equations (10.20) and (10.22), respectively. Then we have the following results:

(1) If $g(t)$ satisfies the hypotheses in I, if $a(t)$, $b(t)$, and $g(t)$ satisfy the equations in (10.34), and if

(10.35) $$a(t) = O(t^{-1}), \quad b(t) = O(t^{-1}),$$

$$a'(t) = O(t^{-2}), \quad b'(t) = O(t^{-2}), \quad a(t) + b(t) = O(t^{-2}),$$

then there exist two solutions of the equation in (10.1) having the asymptotic forms

(10.36) $$u(t) = (1 + o(1))tr(t)$$

and

(10.37) $$u(t) = (1 + o(1))r(t),$$

respectively.

(2) If $g(t)$ has the form in II, if $a(t)$, $b(t)$, and $g(t)$ satisfy the equations in (10.34), and if

$$(10.35) \quad a(t) = O(t^{-1}), \quad b(t) = O(t^{-1}),$$

$$a'(t) = O(t^{-2}), \quad b'(t) = O(t^{-2}), \quad a(t) + b(t) = O(t^{-2}),$$

then there exist two solutions of the equation in (10.1) of the form

$$(10.38) \quad u(t) = (1 + o(1))r(t)|g(t)|^{-1/4} \cdot \exp\left\{\pm \int^t \left(1 + \frac{1}{4c_0}\right)^{1/2} g(t_1)^{1/2} dt_1\right\} \quad \text{if} \quad c_0 \neq -\frac{1}{4}$$

or

$$(10.39) \quad \left.\begin{aligned} u(t) &= (1 + o(1))|g(t)|^{-1/4} r(t) \log t \\ u(t) &= (1 + o(1))|g(t)|^{-1/4} r(t) \end{aligned}\right\} \quad \text{if} \quad c_0 = -\frac{1}{4}.$$

(3) If $g(t)$ satisfies the hypotheses in III, if $a(t)$, $b(t)$ and $g(t)$ satisfy the equations in (10.34), and if

$$(10.40) \quad a(t) = O(|g(t)|^{1/2}), \quad b(t) = O(|g(t)|^{1/2}),$$

$$a'(t) = O(|g(t)|), \quad b'(t) = O(|g(t)|),$$

$$a(t) + b(t) = O(|g(t)|),$$

then there exist two solutions of the equation in (10.1) having the form

(10.41) $$u(t) = (1 + o(1))|g(t)|^{-1/4}r(t)\exp\left\{\pm\int^t g(t_1)^{1/2}dt_1\right\}.$$

(4) If $g(t)$ satisfies the hypotheses in IV, if $a(t)$, $b(t)$ and $g(t)$ satisfy the equations in (10.34), and if

(10.42) $$a(t) = O(|g(t)|),\quad b(t) = O(|g(t)|),$$

$$a'(t) = O(|g(t)|^{3/2}),\quad b'(t) = O(|g(t)|^{3/2}),$$

then there exist two solutions of the equation in (10.1) having the form

(10.43) $$u(t) = (1 + o(1))|g(t)|^{-1/4}r(t)\exp\left\{\pm\int^t g(t_1)^{1/2}dt_1\right\}.$$

The proof of Theorem 6 will be given in §11 below. In the meantime, we wish to illustrate it by discussing the most interesting special case, that in which $a(t)$ and $b(t)$ have asymptotic power series expansions

(10.44) $$a(t) \sim \sum_{n=1}^{\infty} a_n t^{-n},\quad b(t) \sim \sum_{n=1}^{\infty} b_n t^{-n}.$$

Then

(10.45) $$g(t) \sim \sum_{n=1}^{\infty} g_n t^{-n},$$

where

(10.46) $$g_1 = -c_2(a_1 + b_1),$$

$$g_2 = -c_2(a_2+b_2) + \tfrac{1}{2}c_1(a_1+b_1) + b_1 + \tfrac{1}{4}(c_1a_1+c_1b_1-2b_1)^2,$$

etc. Assuming that $a'(t)$, $a''(t)$, $b'(t)$, and $b''(t)$ also have asymptotic power series, we can deduce the asymptotic form of solutions from Theorem 6. First, if $a_1 + b_1 \neq 0$, we use part (4). Here

$$\exp\left\{\pm \int^t g(t_1)^{1/2} dt_1\right\} = \exp\left\{\pm 2(g_1 t)^{1/2}\right\} \cdot \left\{c + O(t^{-1/2})\right\}.$$

Also

$$r(t) = t^r \left\{c + O(t^{-1})\right\},$$

where

$$(10.47) \qquad r = -\frac{1}{2}(c_1 a_1 + c_1 b_1 - c_2 \omega b_1) = \frac{2b_1 - a_1}{3} .$$

Hence, there are two solutions of the form

$$(10.48) \qquad u(t) = (1 + o(1)) t^{r+1/4} \exp\left\{\pm 2(g_1 t)^{1/2}\right\}.$$

On the other hand, if $a_1 + b_1 = 0$, we have

$$g_2 = -c_2(a_2 + b_2) + b_1 + b_1^2.$$

If $g_2 \neq 0$, we apply part (2) of the theorem. Since now $r = b_1$, we obtain, from the equation in (10.38),

$$(10.49) \qquad u(t) = (1 + o(1)) t^{b_1 + 1/2 \pm \alpha}, \quad g_2 \neq -\frac{1}{4},$$

where

$$(10.50) \qquad \alpha = (g_2 + \frac{1}{4})^{1/2} .$$

If $g_2 = -1/4$, we obtain, from the equation in (10.39),

$$(10.51)\qquad u(t) = (1 + o(1))t^{b_1+1/2}\log t \quad \text{and} \quad u(t) = (1 + o(1))t^{b_1+1/2}.$$

Finally, if $g_1 = g_2 = 0$, it follows from part (1) of Theorem 6 that there are two solutions of the form

$$(10.52)\qquad u(t) = (1 + o(1))t^{1+b_1}, \quad u(t) = (1 + o(1))t^{b_1}.$$

It will be clear from the proof in the next section that further terms in these asymptotic expansions can be obtained by using the techniques of §5. This will be discussed further in §12. In the meantime, let us state in a theorem the results obtained here.

Theorem 7. Suppose that the principal root of $h(s) = s + a_0 + b_0e^{-\omega s}$ *lies at* $s = 0$ *and is a double root. Suppose that* $a(t)$ *and* $b(t)$ *have asymptotic power series expansions*

$$a(t) \sim \sum_{n=1}^{\infty} a_n t^{-n}, \quad b(t) \sim \sum_{n=1}^{\infty} b_1 t^{-n},$$

and that $a'(t)$, $b'(t)$, $a''(t)$, *and* $b''(t)$ *exist and have asymptotic power series expansions. Let*

$$g_1 = -2\omega^{-1}(a_1 + b_1),$$

$$g_2 = -2\omega^{-1}(a_2 + b_2) + \tfrac{1}{3}(a_1 + b_1) + b_1$$

$$+ \tfrac{1}{4}(\tfrac{2}{3}a_1 + \tfrac{2}{3}b_1 - 2b_1)^2,$$

$$r = \frac{1}{3}(2b_1 - a_1),$$

$$\alpha = \left(g_2 + \frac{1}{4}\right)^{1/2}.$$

Then the equation in (10.1) has solutions with the following asymptotic forms:

$$u(t) = (1 + o(1))t^{r+1/4}\exp\left\{\pm\ 2(g_1 t)^{1/2}\right\}, \quad g_1 \neq 0,$$

$$u(t) = (1 + o(1))t^{b_1+1/2\pm\alpha}, \quad g_1 = 0,\ g_2 \neq 0,\ g_2 \neq -\frac{1}{4},$$

$$\left.\begin{aligned} u(t) &= (1 + o(1))t^{b_1+1/2}\log t \\ u(t) &= (1 + o(1))t^{b_1+1/2} \end{aligned}\right\}, \quad g_1 = 0,\ g_2 = -\frac{1}{4},$$

$$\left.\begin{aligned} u(t) &= (1 + o(1))t^{b_1+1} \\ u(t) &= (1 + o(1))t^{b_1} \end{aligned}\right\}, \quad g_1 = 0,\ g_2 = 0.$$

It should be kept in mind that in Theorems 6 and 7 we have supposed that the double root has been translated to the origin.

11. Proof of Theorem on Multiple Principal Root

We shall begin by proving part (4) of the theorem, since this is the most difficult in the sense that it is the only part of the proof in which the expansion in (10.13) is essential. The discussion in §8 indicates that we should make a Liouville transformation, $s = s(t)$, where

(11.1) $s(t) = \int^t |g(t_1)|^{1/2} dt_1 .$

This introduces a new real variable s (not the complex variable in (10.2)). It is evident from the relations in (10.33) that $s \to +\infty$ as $t \to +\infty$. The equation in (10.21) is transformed into

$$(11.2) \qquad \frac{d^2x}{ds^2} \pm \frac{g'(t)}{2|g(t)|^{3/2}} \frac{dx}{ds} \mp x = \frac{p_1(t) + v_1(t)}{r(t)|g(t)|}$$

where the upper signs are to be used if $g(t) > 0$ and the lower signs if $g(t) < 0$. (By hypothesis, $g(t) \neq 0$ for $t \geq t_0$). The term containing dx/ds is now eliminated by the substitution

(11.3) $x = y(s)|g(t)|^{-1/4}.$

The resulting equation is

$$(11.4) \qquad \frac{d^2y}{ds^2} + [\mp 1 + g_1(t)] y = \frac{p_1(t) + v_1(t)}{r(t)|g(t)|^{3/4}}$$

where

$$(11.5) \qquad g_1(t) = \frac{5}{16} \frac{|g'(t)|^2}{|g(t)|^3} \mp \frac{g''(t)}{4|g(t)|^2} .$$

If $g(t) > 0$ for $t \geq t_0$, it follows from Lemma 11 that every solution of the equation in (11.4) satisfies an integral equation

$$(11.6)\qquad y(s) = c_4 e^{s} + c_5 e^{-s} + \frac{1}{2}\int_{s_0}^{s}\left[e^{s-s_1} - e^{-(s-s_1)}\right]\left[g_1(t_1)y(s_1) - \frac{p_1(t_1) + v_1(t_1)}{r(t_1)g(t_1)^{3/4}}\right]ds_1,$$

where s_1 and t_1 are related as in (11.1). If $g(t) < 0$ for $t \geq t_0$, we have, instead of (11.6),

$$(11.7)\qquad y(s) = c_4 e^{is} + c_5 e^{-is} + \frac{1}{2i}\int_{s_0}^{s}\left[e^{i(s-s_1)} - e^{-i(s-s_1)}\right]\left[g_1(t_1)y(s_1) - \frac{p_1(t_1) + v_1(t_1)}{r(t_1)|g(t_1)|^{3/4}}\right]ds_1.$$

Since the discussion of the equation in (11.6) and that of the equation in (11.7) are very similar, it will be enough to give only the former. If we return to the original variables by means of the substitutions in (11.3), (11.1), (10.19), and (10.15), we obtain in place of (11.6) the equation

$$(11.8)\qquad u(t) = p(t) + c_4 r(t)g(t)^{-1/4}e^{s(t)} + c_5 r(t)g(t)^{-1/4}e^{-s(t)} + r(t)g(t)^{-1/4}e^{s(t)}\int_{t_0}^{t} e^{-s(t_1)}n(t_1)dt_1 - r(t)g(t)^{-1/4}e^{-s(t)}\int_{t_0}^{t} e^{s(t_1)}n(t_1)dt_1,$$

where, for the sake of abbreviation, we have let

$$(11.9)\qquad n(t) = \frac{g(t)^{3/4}g_1(t)}{2r(t)}\{u(t) - p(t)\} - \frac{p_1(t) + v_1(t)}{2r(t)g(t)^{1/4}}.$$

Here $s(t)$ represents the function defined in (11.1).

We propose to use the equation in (11.8) to show that there are solutions for which

$$u(t) = r(t)g(t)^{-1/4}e^{\pm s(t)}(1 + o(1)),$$

as stated in Theorem 6. The procedure to be used is similar to that in §4, but more complicated in detail. We begin by deriving an expression for $v(t)$ from (11.8). If we let

$$(11.10)\qquad q_{\pm}(t) = r(t)g(t)^{-1/4}e^{\pm s(t)},$$

$$(11.11)\qquad \Delta q_{\pm}(t) = r(t)g(t)^{-1/4}e^{\pm s(t)} - r(t-\omega)g(t-\omega)^{-1/4}e^{\pm s(t-\omega)},$$

we obtain

$$\begin{aligned}(11.12)\qquad v(t) = {} & \Delta p(t) + c_4\Delta q_+(t) + c_5\Delta q_-(t) \\ & + \Delta q_+(t)\int_{t_0}^{t-\omega} e^{-s(t_1)}n(t_1)dt_1 \\ & + q_+(t)\int_{t-\omega}^{t} e^{-s(t_1)}n(t_1)dt_1 \\ & - \Delta q_-(t)\int_{t_0}^{t-\omega} e^{s(t_1)}n(t_1)dt_1 \\ & - q_-(t)\int_{t-\omega}^{t} e^{s(t_1)}n(t_1)dt_1.\end{aligned}$$

Also

$$(11.13) \quad u'(t) = p'(t) + c_4 q'_+(t) + c_5 q'_-(t) + q'_+(t)\int_{t_0}^{t} e^{-s(t_1)} n(t_1)dt_1$$

$$- q'_-(t)\int_{t_0}^{t} e^{s(t_1)} n(t_1)dt_1 .$$

From these expressions we get

$$(11.14) \quad - u_1(t) = v(t) - \omega u'(t)$$

$$= \Delta p(t) - \omega p'(t) + c_4\left[\Delta q_+(t) - \omega q'_+(t)\right]$$

$$+ c_5\left[\Delta q_-(t) - \omega q'_-(t)\right]$$

$$+ \left[\Delta q_+(t) - \omega q'_+(t)\right]\int_{t_0}^{t-\omega} e^{-s(t_1)} n(t_1)dt_1$$

$$+ \left[q_+(t) - \omega q'_+(t)\right]\int_{t-\omega}^{t} e^{-s(t_1)} n(t_1)dt_1$$

$$- \left[\Delta q_-(t) - \omega q'_-(t)\right]\int_{t_0}^{t-\omega} e^{s(t_1)} n(t_1)dt_1$$

$$-\left[q_-(t) - \omega q'_-(t)\right]\int_{t-\omega}^{t} e^{s(t_1)} n(t_1)dt_1 .$$

Finally, from the equation in (11.13) we get

$$(11.15)\quad v'(t) = \Delta p'(t) + c_4 \Delta q_+'(t) + c_5 \Delta q_-'(t)$$
$$+ \Delta q_+'(t) \int_{t_0}^{t-\omega} e^{-s(t_1)} n(t_1) dt_1$$
$$+ q_+'(t) \int_{t-\omega}^{t} e^{-s(t_1)} n(t_1) dt_1$$
$$- \Delta q_-'(t) \int_{t_0}^{t-\omega} e^{s(t_1)} n(t_1) dt_1$$
$$- q_-'(t) \int_{t-\omega}^{t} e^{s(t_1)} n(t_1) dt_1 .$$

Since $s'(t) = g(t)^{1/2}$, $r'(t) = 1/2\left[c_1 a(t) + c_2(1 - \omega) b(t)\right] r(t)$,

$$(11.16)\quad g_{\pm}'(t) = q_{\pm}(t)\left[-\frac{c_1 a(t) + c_2(1-\omega) b(t)}{2} - \frac{g'(t)}{4g(t)} \pm g(t)^{1/2}\right].$$

From the hypotheses in (10.32) and (10.42), it follows that

$$(11.17)\quad q_{\pm}'(t) = g_{\pm}(t) O(g^{1/2}).$$

Moreover, by the Mean Value Theorem,

$$\Delta q_{\pm}(t) = -\omega q_{\pm}'(t - \omega\ell) = q_{\pm}(t - \omega\ell) O(g(t - \omega\ell)^{1/2}),$$
$$0 < \ell < 1.$$

Since $a(t)$, $b(t)$, and $g(t)$ all satisfy the relations in (10.34),

$$(11.18)\quad \Delta q_{\pm}(t) = q_{\pm}(t) O(g(t)^{1/2}).$$

Since $a' = O(g)$, $b' = O(g)$, $g'' = O(g^2)$, and $g' = O(g^{3/2})$, it can be deduced from (11.16) that

$$q''_{\pm}(t) = q_{\pm}(t)O(g). \tag{11.19}$$

Hence

$$\Delta q'_{\pm}(t) = q_{\pm}(t)O(g), \tag{11.20}$$

and

$$\Delta \dot{q}_{\pm}(t) - \omega q'_{\pm}(t) = q_{\pm}(t)O(g). \tag{11.21}$$

From the definition of $p(t)$ in (10.9), and the inequality in (10.10), we find that

$$|p(t)| \leq ce^{kt} \int_{t_0}^{t} \Big\{ |a(t_1) + b(t_1)| \, |u(t_1)| + |b(t_1)| \, |v(t_1)| \Big\} e^{-kt_1} dt_1.$$

Let

$$m_1(t) = \max_{t_0 \leq t_1 \leq t} \left| \frac{u(t_1)}{q_+(t_1)} \right|, \tag{11.22}$$

$$m_2(t) = \max_{t_0 \leq t_1 \leq t} \left| \frac{v(t_1)}{q_+(t_1) g(t_1)^{1/2}} \right| .$$

Then since $|a + b|$ and $|b|$ are $O(g)$,

$$|p(t)| \leq ce^{kt+s(t)}\left\{m_1(t) + m_2(t)\right\}\int_{t_0}^{t} g(t_1)^{3/4} e^{-kt_1} r(t_1)dt_1 .$$

The integral is of the form in Lemma 3, with $g(t)^{3/4}$ in place of $g(t)$ and

$$r(t) = -\frac{1}{2}\left[c_1 a(t) + c_2(1 - \omega)b(t)\right].$$

From the hypotheses in IV and (4), it can be verified that the conditions in Lemma 3 are met. Hence

$$(11.23) \qquad |p(t)| \leq ce^{s(t)}g(t)^{3/4}r(t)\left\{m_1(t) + m_2(t)\right\}$$

$$= cq_+(t)g(t)\left\{m_1(t) + m_2(t)\right\}.$$

By differentiating the equation in (10.9), we also find that

$$(11.24) \qquad |p'(t)| \leq cq_+(t)g(t)\left\{m_1(t) + m_2(t)\right\}$$

and

$$(11.25) \qquad |\Delta p(t)| \leq cq_+(t)g(t)\left\{m_1(t) + m_2(t)\right\}.$$

Hence

$$(11.26) \qquad |p_1(t)| \leq cq_+(t)g(t)^2\left\{m_1(t) + m_2(t)\right\}.$$

Since

$$(11.27) \qquad |g_1(t)| \leq cg(t),$$

by the hypotheses in (10.32), we can deduce from the equation in (11.9) and the inequalities in (11.23) and (11.26) that

$$(11.28)\qquad |n(t)| \leq cg(t)^{3/2}e^{s(t)}\{m_1(t) + m_2(t)\} + c\left|\frac{v_1(t)}{r(t)g(t)^{1/4}}\right|.$$

Using this inequality in (11.8), we get

$$\left|\frac{u(t)}{q_+(t)}\right| \leq cg(t)\{m_1(t) + m_2(t)\} + c\{m_1(t) + m_2(t)\}\int_{t_0}^{t} g(t_1)^{3/2}dt_1 + c\int_{t_0}^{t}\left|\frac{v_1(t_1)e^{-s(t_1)}}{r(t_1)g(t_1)^{1/4}}\right|dt_1.$$

Since $\int_{t_0}^{\infty} g(t_1)^{3/2}dt_1$ is as small as we please if t_0 is sufficiently large, and $g(t) = o(1)$, it follows that

$$(11.29)\qquad m_1(t) \leq c + cm_2(t) + c\int_{t_0}^{t}\left|\frac{v_1(t_1)e^{-s(t_1)}}{r(t_1)g(t_1)^{1/4}}\right|dt_1.$$

On the other hand, from (11.12) and (11.18), we have

$$\left|\frac{v(t)}{q_+(t)}\right| \leq cg(t)\{m_1(t) + m_2(t)\} + cg(t)^{1/2} + cg(t)^{1/2}\int_{t_0}^{t} e^{-s(t_1)}|n(t_1)|\,dt_1 + 2\int_{t-\omega}^{t} e^{-s(t_1)}|n(t_1)|\,dt_1.$$

Using (11.28) and (11.29) and the fact that

$$\int_{t-\omega}^{t} g(t_1)^{3/2}dt_1 = O(g^{3/2}),$$

and supposing t_0 sufficiently large, it follows that

$$(11.30)\qquad m_2(t) \leq c + c\int_{t_0}^{t}\left|\frac{v_1 e^{-s(t_1)}}{rg^{1/4}}\right|dt_1 + c\int_{t-\omega}^{t}\left|\frac{v_1 e^{-s(t_1)}}{rg^{3/4}}\right|dt_1 .$$

Hence also

$$(11.31)\qquad m_1(t) \leq c + c\int_{t_0}^{t}\left|\frac{v_1 e^{-s(t_1)}}{rg^{1/4}}\right|dt_1 + c\int_{t-\omega}^{t}\left|\frac{v_1 e^{-s(t_1)}}{rg^{3/4}}\right|dt_1 .$$

In the same way, we find from the equation in (11.15) that

$$(11.32)\qquad \left|\frac{v'(t)}{q_+(t)g(t)}\right| \leq c + c\int_{t_0}^{t}\left|\frac{v_1 e^{-s(t_1)}}{rg^{1/4}}\right|dt_1 + c\int_{t-\omega}^{t}\left|\frac{v_1 e^{-s(t_1)}}{rg^{3/4}}\right|dt_1 .$$

Since

$$\left| \int_{t-\omega}^{t} e^{-s(t_1)} n(t_1) - \frac{q_-(t)}{q_+(t)} \int_{t-\omega}^{t} e^{s(t_1)} n(t_1) dt_1 \right|$$

$$\leq \left\{ 1 - e^{2s(t-\omega)-2s(t)} \right\} \int_{t-\omega}^{t} e^{-s(t_1)} |n(t_1)| dt_1$$

$$\leq cg(t)^{1/2} \int_{t-\omega}^{t} e^{-s(t_1)} |n(t_1)| dt_1,$$

we find from the equation in (11.14) that

$$(11.33) \quad \left| \frac{u_1(t)}{q_+(t)} \right| \leq cg(t) \left\{ m_1(t) + m_2(t) \right\} + cg(t)$$

$$+ cg(t) \int_{t_0}^{t} e^{-s(t_1)} |n(t_1)| dt_1$$

$$+ cg(t)^{1/2} \int_{t-\omega}^{t} e^{-s(t_1)} |n(t_1)| dt_1.$$

Using the inequality in (11.28), we therefore obtain

$$(11.34) \quad \left| \frac{u_1(t)}{q_+(t)g(t)} \right| \leq c + c \int_{t_0}^{t} \left| \frac{v_1 e^{-s(t_1)}}{rg^{1/4}} \right| dt_1$$

$$+ c \int_{t-\omega}^{t} \left| \frac{v_1 e^{-s(t_1)}}{rg^{3/4}} \right| dt_1.$$

It follows, since $b(t) = O(g)$ and $b'(t) = O(g^{3/2})$, that

$$\left| \frac{v_1(t)}{q_+(t)g(t)^2} \right| \leq c + c \int_{t_0}^{t} \left| \frac{v_1 e^{-s(t_1)}}{rg^{1/4}} \right| dt_1$$

$$+ c \int_{t-\omega}^{t} \left| \frac{v_1 e^{-s(t_1)}}{rg^{3/4}} \right| dt_1.$$

Let

$$(11.35)\qquad m_3(t) = \max_{t_0 \leq t_1 \leq t} \left| \frac{v_1(t_1)}{q_+(t_1)g(t_1)^2} \right| .$$

Then

$$(11.36)\qquad \left| \frac{v_1(t)}{q_+(t)g(t)^2} \right| \leq c + cm_3(t) \int_{t_0}^{t} g(t_1)^{3/2} dt_1$$

$$+ \, cm_3(t) \int_{t-\omega}^{t} g(t_1) dt_1 .$$

If t_0 is sufficiently large, it follows that $m_3(t) \leq c$. Hence,

$$(11.37)\qquad |v_1(t)| \leq cq_+(t)g(t)^2 \leq ce^{s(t)}r(t)g(t)^{7/4},$$

$$(11.38)\qquad |v(t)| \leq cq_+(t)g(t)^{1/2} \leq ce^{s(t)}r(t)g(t)^{1/4},$$

$$(11.39)\qquad |u(t)| \leq cq_+(t) \leq ce^{s(t)}r(t)g(t)^{-1/4}.$$

Once these inequalities have been established, the asymptotic form of $u(t)$ can be found from the equation in (11.8). The inequalities in (11.23) and (11.26) now become

$$|p(t)| \leq ce^{s(t)}g(t)^{3/4}r(t),$$

$$|p_1(t)| \leq ce^{s(t)}g(t)^{7/4}r(t),$$

and that in (11.28) shows that $n(t)e^{-s(t)}$ is absolutely integrable. Therefore

$$(11.40)\qquad u(t) = c_6 r(t)g(t)^{-1/4}e^{s(t)} + p(t) + c_5 r(t)g(t)^{-1/4}e^{-s(t)}$$

$$- r(t)g(t)^{-1/4}e^{s(t)}\int_t^\infty e^{-s(t_1)}n(t_1)dt_1$$

$$- r(t)g(t)^{-1/4}e^{-s(t)}\int_{t_0}^t e^{s(t_1)}n(t_1)dt_1$$

where

$$(11.41)\qquad c_6 = c_4 + \int_{t_0}^\infty e^{-s(t_1)}n(t_1)dt_1 .$$

Taking $c_4 = 1$, $c_5 = 0$, and t_0 sufficiently large, we see that $c_6 \neq 0$. Using Lemma 7, with $n = 3/2$, we find that

$$\int_{t_0}^t e^{s(t_1)}|n(t_1)|dt_1 \leq c\int_{t_0}^t g(t_1)^{3/2}e^{2s(t_1)}dt_1$$

$$\leq cg(t)e^{2s(t)}.$$

Therefore

$$u(t) = (c_6 + o(1))r(t)g(t)^{-1/4}e^{s(t)}.$$

To establish the existence of a solution of the form

$$(c + o(1))r(t)g(t)^{-1/4}e^{-s(t)},$$

we use the method of successive approximations, as in §6. The equation in (11.8) is first replaced by

$$(11.42)\quad u(t) = p(t) + r(t)g(t)^{-1/4}e^{-s(t)}$$

$$- r(t)g(t)^{-1/4}e^{s(t)}\int_t^{\infty} e^{-s(t_1)}n(t_1)dt_1$$

$$+ r(t)g(t)^{-1/4}e^{-s(t)}\int_t^{\infty} e^{s(t_1)}n(t_1)dt_1 .$$

We shall omit the details of the argument.

As previously remarked, the argument is virtually unchanged if $g(t) < 0$ for $t \geq t_0$. In that case, there e solutions of the form

$$u(t) = (c + o(1))r(t)\,|g(t)|^{-1/4}e^{\pm is(t)} .$$

We have thus completed the proof of part (4) of Theorem 6. The demonstration given needs only very slight modification to apply to part (3) of the theorem. In fact, all the relations in (11.1) - (11.25) are still valid. Since we are now assuming only that $a(t), b(t) = O(g(t)^{1/2})$, the relation in (11.26) becomes

$$(11.43)\quad |r_1(t)| \leq cq_+(t)g(t)^{3/2}\{m_1(t) + m_2(t)\} .$$

It is now found that $|g_1(t)| \leq c$, and the inequality in (11.28) must be replaced by

$$(11.44)\quad |n(t)| \leq ce^{u(t)}\{m_1(t) + m_2(t)\}\{g(t)^{1/2}g_1(t) + g(t)\}$$

$$+ c\left|\frac{v_1(t)}{r(t)g(t)^{1/4}}\right| .$$

From the equation in (11.8), we therefore get

$$\left|\frac{u(t)}{q_+(t)}\right| \leq cg(t)\{m_1(t) + m_2(t)\} + c$$

$$+ c\{m_1(t) + m_2(t)\} \int_{t_0}^{t} \left[g(t_1)^{1/2} g_1(t_1) + g(t_1)\right] dt_1$$

$$+ c \int_{t_0}^{t} \left|\frac{v_1 e^{-s(t_1)}}{rg^{1/4}}\right| dt_1 .$$

Since $g(t)$ and $g(t)^{1/2} g_1(t)$ are integrable over the infinite range, the equation in (11.29) is now obtained. Since

$$(11.45) \quad \int_{t-\infty}^{t} |g(t_1)^{1/2} g_1(t_1) + g(t_1)|\, dt_1$$

$$\leq c\left[g(t)^{1/2} g_1(t) + g(t)\right] = o(g^{1/2})$$

we find in the same way as before that the equation in (11.30) holds. Likewise (11.31) - (11.34) remain true. From the definition of $v_1(t)$ in (10.18), and the hypothesis that $b = O(g^{1/2})$, $b' = O(g)$, we therefore obtain in the same way as before

$$|v_1(t)| \leq cq_+(t) g(t)^{3/2} \leq ce^{s(t)} r(t) g(t)^{5/4}$$

and therefore

$$|u(t)| \leq cq_+(t) \leq ce^{s(t)} r(t) g(t)^{-1/4} .$$

The rest of the proof of part (3) is as before.

We now turn to a discussion of part (2) of the theorem.

We again use the Liouville transformation in (11.1), which has the form

$$s(t) = |c_0|^{1/2} \log t + c + O(t^{-1}), \tag{11.46}$$

and the transformation in (11.3). We find, however, that

$$g_1(t) = -\frac{1}{4|c_0|} + g_2(t) \tag{11.47}$$

where

$$g_2(t) = O(t^{-1}). \tag{11.48}$$

If we put

$$c_3 = \left|1 + \frac{1}{4c_0}\right|^{1/2}, \tag{11.49}$$

the equation becomes

$$\frac{d^2y}{ds^2} \mp c_3^2 y = -g_2(t)y + \frac{p_1(t) + v_1(t)}{r(t)|g(t)|^{3/4}}, \tag{11.50}$$

where the lower sign is to be used is $c_0 < -1/4$, and the upper sign if $c_0 > -1/4$. We can show by the method used to prove part (3) of Theorem 6 that there are solutions of the form

$$u(t) = (1 + o(1))r(t)|g(t)|^{-1/4}e^{\pm c_3 s(t)}, \quad c_0 > -\frac{1}{4}, \tag{11.51}$$

$$u(t) = (1 + o(1))r(t)|g(t)|^{-1/4}e^{\pm ic_3 s(t)}, \quad c_0 < -\frac{1}{4}.$$

Only minor modifications in the proof are required. The two equations in (11.51) can be combined in the single relation

$$(11.52)\qquad u(t) = (1 + o(1))r(t)|g(t)|^{-1/4}$$
$$\cdot \exp\left\{\pm\int^{t} \left(1 + \frac{1}{4c_0}\right)^{1/2} g(t_1)^{1/2}dt_1\right\}, \quad c_0 \neq -\frac{1}{4}.$$

On the other hand, if $c_0 = -1/4$, then $c_3 = 0$, and the equation in (11.50) leads to a representation of the form

$$(11.53)\qquad u(t) = p(t) + c_4 g(t)^{-1/4}r(t) + c_5 g(t)^{-1/4}r(t)s(t)$$
$$+ g(t)^{-1/4}r(t)s(t)\int_{t_0}^{t} n(t_1)dt_1$$
$$- g(t)^{-1/4}r(t)\int_{t_0}^{t} s(t_1)n(t_1)dt_1$$

where

$$(11.54)\qquad n(t) = \frac{g_2(t)g(t)^{3/4}}{r(t)}\left\{u(t) - p(t)\right\} - \frac{p_1(t) + v_1(t)}{r(t)g(t)^{1/4}},$$

rather than an equation of the form in (11.8). Using the same procedure as before, we can show that one solution has the form

$$u(t) = (1 + o(1))g(t)^{-1/4}r(t)s(t),$$

and, using successive approximations, that a second has the form

$$u(t) = (1 + o(1))g(t)^{-1/4}r(t).$$

This is the result stated in part (2) of Theorem 6.

Finally, to prove part (1) of Theorem 6, we deal directly with the equation in (10.21), as the Liouville transformation is unnecessary. From Lemma 11, we obtain the integral equation

$$(11.55)\qquad u(t) = p(t) + c_4 r(t) + c_5 t r(t) + r(t)\int_{t_0}^{t} (t - t_1) n(t_1)\,dt_1,$$

where c_4 and c_5 are arbitrary and

$$(11.56)\qquad n(t) = -\frac{g(t)}{r(t)}\left\{u(t) - p(t)\right\} - \frac{p_1(t) + v_1(t)}{r(t)}\,.$$

We now proceed in the same way as before, obtaining expressions for $v(t)$, $u'(t)$, $v'(t)$, and $u_1(t)$ from the equation in (11.55). Since

$$|\Delta(tr(t))| \leq cr(t),$$

$$|\Delta r(t)| \leq c\,\frac{r(t)}{t},$$

etc., we can establish the following inequalities by following the same procedure we used for the equation in (11.8):

$$|u(t)| \leq ctr(t),$$

$$|v(t)| \leq cr(t).$$

Hence

$$|p(t)| \leq cr(t)/t,$$

$$|n(t)| \leq ctg(t) + ct^{-2}.$$

If $c_5 \neq 0$, it follows from the equation in (11.55) that

$$u(t) = p(t) + c_6 tr(t) + c_4 r(t) - tr(t)\int_t^{\infty} n(t_1)dt_1$$

$$- r(t)\int_{t_0}^{t} t_1 n(t_1)dt_1,$$

where

$$c_6 = c_5 + \int_{t_0}^{\infty} n(t_1)dt_1.$$

Taking $c_5 = 1$, $c_4 = 0$, and choosing t_0 so large that $c_6 \neq 0$, we obtain a solution $u(t) = tr(t)(1 + o(1))$. To show the existence of a solution of the form $u(t) = r(t)(1 + o(1))$, we replace the equation in (11.55) by

$$(11.57)\qquad u(t) = p(t) + r(t) - tr(t)\int_t^{\infty} n(t_1)dt_1$$

$$+ r(t)\int_t^{\infty} t_1 n(t_1)dt_1,$$

and use successive approximations. In this way, the proof of Theorem 6 can be completed.

12. Other Multiple Roots

In §§10-11, we have given a complete discussion of the differential-difference equation

$$(12.1) \qquad u'(t) + (a_0 + a(t))u(t) + (b_0 + b(t))u(t - \omega) = 0$$

having a principal double root. In this section, we should like to give a brief indication of how to handle other multiple roots. In the first place, suppose the equation has a real double root, not necessarily the principal root, at $s = \lambda$. Again we can suppose this root has been translated to $s = 0$. We establish the existence of solutions having the form indicated in Theorems 6 and 7 by use of the method of successive approximations, as we did in §6 for a real, simple root. The definition of $p(t)$ in (10.9) must now be replaced by

$$(12.2) \qquad p(t) = -\int_{t_0}^{t} \left[a(t_1)u(t_1) + b(t_1)u(t_1 - \omega)\right] k_1(t - t_1)dt_1$$

$$+\int_{t}^{\infty} \left[a(t_1)u(t_1) + b(t_1)u(t_1 - \omega)\right] k_2(t - t_1)dt_1$$

but the other equations in (10.8) - (10.22) are unaltered in appearance. If the hypotheses of part (4) of Theorem 6 are satisfied, for example, we proceed as in §11, and again obtain the integral equation in (11.8). If we desire to find a solution of the form

$$(12.3) \qquad u(t) = (1 + o(1))r(t)g(t)^{-1/4}e^{s(t)},$$

we define

$$(12.4)\qquad u_{m+1}(t) = p_m(t) + r(t)g(t)^{-1/4}e^{s(t)}$$

$$+ r(t)g(t)^{-1/4}e^{s(t)}\int_{t_0}^{t} e^{-s(t_1)}n_m(t_1)dt_1$$

$$- r(t)g(t)^{-1/4}e^{-s(t)}\int_{t_0}^{t} e^{s(t_1)}n_m(t_1)dt_1,$$

$$t \geq t_0, \quad m = 0,1,2,\ldots,$$

where

$$(12.5)\qquad p_m(t) = -\int_{t_0}^{t}\left[a(t_1)u_m(t_1) + b(t_1)u_m(t_1 - \omega)\right]k_1(t - t_1)dt_1$$

$$+\int_{t}^{\infty}\left[a(t_1)u_m(t_1) + b(t_1)u_m(t_1 - \omega)\right]k_2(t - t_1)dt_1$$

and where

$$(12.6)\qquad n_m(t) = \frac{g(t)^{3/4}g_1(t)}{2r(t)}\left\{u_m(t) - p_m(t)\right\}$$

$$- \frac{p_{1,m}(t) + v_{1,m}(t)}{2r(t)g(t)^{1/4}}.$$

There is now no difficulty in deriving expressions for $v_m(t)$, $u_m'(t)$, $u_{1,m}(t)$, $v_m'(t)$ like those in (11.12) - (11.15), respectively. Proceeding by induction, we can show that $u_m(t)r(t)^{-1}g(t)^{1/4}e^{-s(t)}$ is bounded, and then that the sequence $\{u_m(t)\}$ converges, uniformly in any finite interval, to a solution of the equations in (12.4) and (12.1). We shall omit the details. That $u(t)$ has the form in (12.3)

follows easily from the integral equation satisfied by $u(t)$. In order to establish the existence of a solution of the form

$$(12.7) \qquad u(t) = (1 + o(1))r(t)g(t)^{-1/4}e^{-s(t)},$$

we replace the equation in (12.4) by

$$(12.8) \qquad u_{m+1}(t) = p_m(t) - r(t)g(t)^{-1/4}e^{s(t)}\int_t^{\infty} e^{-s(t_1)}n_m(t_1)dt_1$$

$$+ r(t)g(t)^{-1/4}e^{-s(t)}$$

$$+ r(t)g(t)^{-1/4}e^{-s(t)}\int_t^{\infty} e^{s(t_1)}n_m(t_1)dt_1$$

and proceed as before. The other parts of Theorem 6 can be dealt with similarly. Hence, <u>Theorems 6 and 7 remain valid if</u> $s = 0$ <u>is any double root, not necessarily the principal root</u>.

We should again like to remark that if

$$(12.9) \qquad a(t) \sim \sum_{n=1}^{\infty} a_n t^{-n}, \quad b(t) \sim \sum_{n=1}^{\infty} b_n t^{-n},$$

then the solutions $u(t)$ of the equation in (12.1) have full asymptotic expansions, which can be found in a fashion similar to that in §5. Suppose, for example, that $a_1 + b_1 \neq 0$, $g_1 > 0$, and that

$$u(t) = (1 + o(1))t^{r+1/4} \exp\left\{2(g_1 t)^{1/2}\right\}.$$

This function satisfies an equation of the form in (11.8). By the relations in §11, we have

$$|n(t)| \leq ct^{-3/2} \exp\left\{2(g_1 t)^{1/2}\right\}.$$

The equation in (11.8) can therefore be rewritten as

$$\begin{aligned}(12.10) \qquad u(t) = p(t) &+ c_6 t^{r+1/4} \exp\left\{2(g_1 t)^{1/2}\right\} \\ &+ c_5 t^{r+1/4} \exp\left\{-2(g_1 t)^{1/2}\right\} \\ &- t^{r+1/4} \exp\left\{2(g_1 t)^{1/2}\right\} \int_t^\infty n(t_1) \exp\left\{-2(g_1 t_1)^{1/2}\right\} dt_1 \\ &- t^{r+1/4} \exp\left\{-2(g_1 t)^{1/2}\right\} \int_{t_0}^t n(t_1) \exp\left\{2(g_1 t_1)^{1/2}\right\} dt_1 .\end{aligned}$$

It follows that

$$(12.11) \qquad u(t) = \left[c_6 + O(t^{-1/2})\right] t^{r+1/4} \exp\left\{2(g_1 t)^{1/2}\right\}.$$

With the aid of this expression, we can demonstrate that

$$p(t) = \left[c_7 t^{-1} + O(t^{-3/2})\right] t^{r+1/4} \exp\left\{2(g_1 t)^{1/2}\right\},$$

using the technique in §5. Then

$$p_1(t) = \left[c_8 t^{-2} + O(t^{-5/2})\right] t^{r+1/4} \exp\left\{2(g_1 t)^{1/2}\right\}.$$

Directly from the relation in (12.11) we can find similar asymptotic expressions for $v(t)$, $v'(t)$, $u_1(t)$, $v_1(t)$, and finally deduce that

$$n(t) = \left[c_9 t^{-3/2} + O(t^{-2})\right] \exp\left\{2(g_1 t)^{1/2}\right\}.$$

Using this relation in (12.10), we get

$$u(t) = \left[c_6 - 2c_9 t^{-1/2} + O(t^{-1})\right] t^{r+1/4} \exp\left\{2(g_1 t)^{1/2}\right\}.$$

Repetitions of this argument show that the solution has an asymptotic expansion

$$(12.12) \qquad u(t) \sim t^{r+1/4} \exp\left\{2(g_1 t)^{1/2}\right\} \sum_{n=0}^{\infty} u_n t^{-n/2} \quad (u_0 \neq 0).$$

Once it is known that the solution has an expansion of the indicated form, the values of the coefficients u_n and of the parameters r and g_1 are most easily found by substituting the expression for $u(t)$ into the original differential-difference equation, and equating coefficients of like powers of t.

Asymptotic expansions of $u(t)$ can be found in a similar way in the other cases of Theorem 7. These expressions may involve combinations of powers of t and $\log t$. For example, if $g_1 = g_2 = 0$, one solution has the form

$$u(t) \sim t^{1+b_1}\left[u_0 + \sum_{n=1}^{\infty} \frac{u_n + u_n' \log t}{t^n}\right].$$

In some of the other cases, higher powers of $\log t$ appear.

We shall close this section with some remarks about equations of more complicated form than that in (12.1). For equations of higher order (in derivatives or differences) than the one in (12.1), it is possible to have non-real double roots, or roots of multiplicity greater than two. In the

former situation, we proceed as in §7, while in the latter, we have to deal with higher order differential equations. For example, if a root has multiplicity three, we are led to an equation of the form

$$x'''(t) + c_1(t)x'(t) + c_2(t)x(t) = w(t).$$

Solutions of such an equation can be written in terms of integral operators by using the method of variation of parameters. If necessary, we first make a Liouville transformation $s = s(t)$,

$$s(t) = \int^t \lambda(t_1)dt_1$$

where $\lambda(t)$ is one root of the characteristic equation

$$\lambda^3 + c_1(t)\lambda + c_2(t) = 0.$$

As the details become even more involved than before, we shall omit a more complete discussion.

13. More General Functional Equations

In attempting to extend the foregoing results to more general linear functional equations of the form

(13.1) $\quad L(u) = g(t)u,$

where

(13.2) $\quad g(t) \sim \frac{c_1}{t} + \frac{c_2}{t^2} + \cdots,$

we see that the techniques we have employed hinge upon two basic facts:

(13.3) (a) The solution of $L(u) = f(t)$, with appropriate initial conditions, has the form

$$u = \int^{t} k(t - s)f(s)ds.$$

(b) $k(t) \sim \sum_{k=1}^{\infty} a_k e^{\lambda_k t}$, $Re(\lambda_1) \geq Re(\lambda_2) > \cdots$,

as $t \to \infty$.

Once it has been established that the linear operator L possesses the required properties, we may readily obtain analogues of the foregoing results.

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